math099.0 High school mathematics

Many

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Efnisyfirlit

1 Integer arithmetic

1.1 Integer arithmetic

1.1.1 Handout

Natural numbers

The numbers $1,2,3,4,\ldots$ are called the *natural numbers*, which we denote by $\mathbb{N}$. Two operations are defined on this set, namely *addition* and *multiplication*. Formally, each pair (a,b) of natural numbers a and b is associated with exactly one number $a+b$ which is termed the *sum* of a and b and another number, ab, the *product* of a and b. The product is also commonly denoted $a\cdot b$.
Certain rules of arithmetic apply to these operations.

$(a+b)+c = a+(b+c)$	*(associative rule for addition),*
$(ab)c = a(bc)$	*(associative rule for multiplication),*
$a+b = b+a$	*(commutative law for addition),*
$ab = ba$	*(commutative law for multiplication),*
$a(b+c) = ab+ac$	*(distributive law),*
$1a = a$	*(1 is the multiplicative identity).*

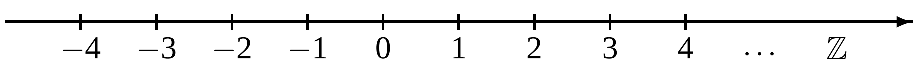
The set $\mathbb{N}$ has an *ordering* so that for any two numbers a and b one of three conditions is satisfied: *a is less than b*, denoted $a<b$, *a is equal to b*, denoted $a=b$ or *a is greater than b*, denoted $a>b$. Formally, these are defined as follows: a is *less than b*, if there is a natural number c such that $a+c=b$, and a is *greater than b*, if b is less than a.
Two important rules apply to the ordering of natural numbers:

If $a<b$ then $a+c<b+c$	*(addition preserves order).*
If $a<b$ then $ac<bc$	*(multiplication preserves order).*

If $a<b$ and $a+c=b$, then the number c is called the *difference* between b and a and we write $c=b-a$. On the other hand if $a>b$ and $a=b+d$, then the number d is the *difference* between a and b and we write $d=a-b$.
Computation with natural numbers is imperfect, since it is for example not always possible to perform *subtraction*, i.e. to find a natural number x such that $a=b+x$. This is only possible if $a>b$. The number x is then called the *difference* between the numbers a and b and is denoted by $a-b$.
To get past this the set of natural numbers is extended by first adding the number 0, called *zero* and this is normally interpreted as the starting point of the natural numbers on the line and then the numbers $-1,-2,-3,-4,\ldots$ are added to the number system. This enlarged number system is called the *integers* and is denoted by $\mathbb{Z}$.
On $\mathbb{Z}$ we can define addition and multiplication. These operations are subject to the same rules as apply to the natural numbers. Formally this is done in such a way as to make 0 an *additive identity*, which means that

$$a+0=a, \qquad a\in\mathbb{Z}.$$

In addition we now obtain the result that every number has an *additive inverse*, which means that for every $a\in\mathbb{Z}$ there is a $b\in\mathbb{Z}$, such that $a+b=0$. The additive inverse is denoted by $-a$.
As before we depict the numbers by laying them out on a number line.



Integer division
If a and b are integers and $b \neq 0$, and there is an integer x such that $a = bx$, then we say that *a is divisible by b*. If such a number x exists, then it is called the *ratio* of the numbers a and b and is denoted $\frac{a}{b}$ or a/b. The operation of finding the number x is called *division*.
It is not possible for all numbers a and b with $b \neq 0$ to divide a by b, i.e. to find an integer x such that $a = bx$. On the other hand it is always possible to find integers q and r so that $a = bq + r$ and $0 < q < b$. This operation is called *division with remainder*. In this context, q is the *quotient* and r is the *remainder* of the division of a by b.
Example 1.1.1
Divide 498 by 7 with remainder.
Solution:
$$\frac{498}{7} = \frac{4}{7} \cdot 100 + \frac{98}{7} = \frac{49}{7} \cdot 10 + \frac{8}{7} = 70 + \frac{8}{7} = 71 + \frac{1}{7}$$
so 7 divides 498 71 times with the remainder 1.

2 Prime numbers

2.1 Prime numbers

2.1.1 Handout

Prime numbers and prime factorization

A natural number $a \in \mathbb{N}$ is said to be *divisible* by the number $b \in \mathbb{N}$ if there is a number $c \in \mathbb{N}$ such that $a = bc$. Any number a is divisible by 1 and itself since $a = 1 \cdot a$. A natural number ≥ 2 which is only divisible by 1 and itself is called *prime*. The first primes are

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, \ldots.$$

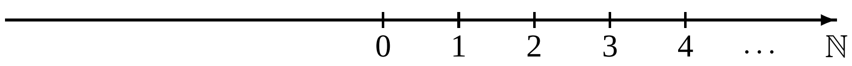
Any natural number $a \geq 2$ can be written as a product of primes

$$a = p_1 p_2 p_3 \cdots p_m$$

where the primes p_j may be repeated. For example,

$$7 = 7, \quad 24 = 2 \cdot 2 \cdot 2 \cdot 3 = 2^3 \cdot 3, \quad 250 = 2 \cdot 5 \cdot 5 \cdot 5 = 2 \cdot 5^3.$$

A factorization of natural numbers into primes is termed a *prime factorization*.
Visually, the natural numbers can be depicted as evenly distributed along a line.



We select an initial point of reference, 0, and place this on the line. Subsequently we select a unit length for the line and mark a point one unit length to the right of 0 and label this with 1. This is continued according to the diagram. It follows that we will have $a < b$ if and only if b is to the right of a on the line.
The operations on $\mathbb{N}$ can now be described as moves along the line. The operation of adding 1 to a number m corresponding to a point on the line is equivalent to taking one move to the right along the line to the point corresponding to $m + 1$. If the number n is added to m then

this operation in merely repeated n times. Multiplication is described in the same manner, the product mn is defined as $m+m+\ldots+m$ where there are a total of n terms in the sum. In this manner, mn corresponds to n moves of the length from 0 to the point m.
Normally one does not distinguish between the number $n \in \mathbb{N}$ and the point on the line which corresponds to n.

Induction

Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ denote the set of natural numbers and the number 0. The following *well-ordering* axiom applies to this set:

Any non-empty subset of $\mathbb{N}_0$ has a smallest element.

The *induction* theorem is a consequence of the well-ordering axiom:

Let $a \in \mathbb{N}_0$, $p(n)$ be a statement about $n \in \mathbb{N}_0$ and assume that the following conditions hold:

(1) The statement $p(a)$ is true.

(2) If $q \in \mathbb{N}_0$, $q \geq a$ and it is assumed that $p(q)$ is true then it follows that the statement $p(q+1)$ is also true.

Then $p(n)$ is true fall all $n \geq a$.

A proof which uses the induction theorem is called a *proof by induction*.
Proof by induction can be used to show that the equation

$$\sum_{k=1}^{n} k = 1+2+3+\cdots+n = \frac{1}{2}n(n+1)$$

holds for all natural numbers n. Let the statement $p(n)$ be that this equation holds for a particular n. Consider first $n = 1$. In this case there is only one term in the sum and that term is 1. The other side of the equality sign contains the expression $\frac{1}{2} \cdot 1(1+1) = 1$. Hence the statement $p(1)$ is true.
Now assume that $q \in \mathbb{N}$ and that $p(q)$ is true i.e. that the equation $\sum_{k=1}^{q} k = \frac{1}{2}q(q+1)$ holds. Using this assumption we obtain the following

$$\begin{aligned}
\sum_{k=1}^{q+1} k &= \left(\sum_{k=1}^{q} k\right) + q + 1 = \frac{1}{2} \cdot q(q+1) + q + 1 \\
&= \frac{1}{2}(q(q+1) + 2(q+1)) = \frac{1}{2}(q+1)(q+2) \\
&= \frac{1}{2}(q+1)((q+1)+1),
\end{aligned}$$

which implies that $p(q+1)$ is true. We have therefore shown that the two conditions of the induction theorem hold and the theorem thus implies that the equation holds for all natural numbers n.

Example 2.1.1

a) Prime factorize 273.
Solution: We see that 2 does not divide 273 but $3 \cdot 91 = 297$. We know that 3 is a prime number. Then we check 91. 5 does not divide 91 but $91 = 7 \cdot 13$. 7 and 13 are prime numbers so 273 prime factorized is $3 \cdot 7 \cdot 13$.
b) Prime factorize 101.
Solution: We start by checking the lowest prime numbers. 2 does not divide 101, $101 = 3 \cdot 33 + 2$ so 3 does not divide 101 . 5 divides 100 and thus not 101. 7 does not divide 101 either. The next prime number is 11 but $11^2 = 121 > 101$ so if any prime larger than 7 divides 101 some prime smaller than 11 also has to divide it but we have ruled that possibility out so we know that 101 is a prime number and the number itself is it's prime factorization.

3 Fractions

3.1 Fractions

3.1.1 Handout

Rational numbers (fractions)

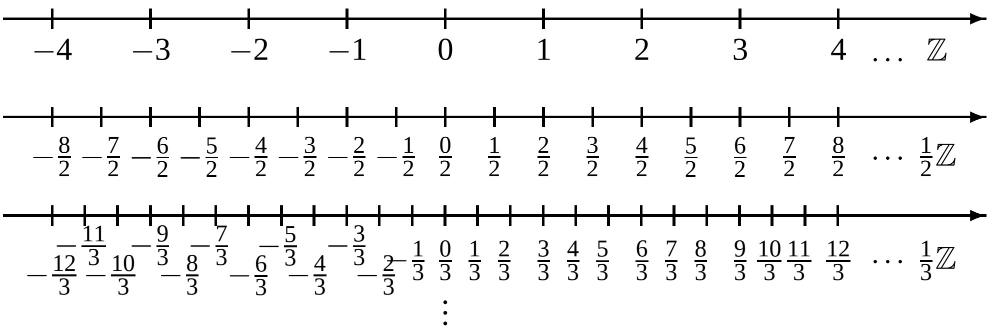
Arithmetic with integers is imperfect in part because it is not always possible to conduct division except with a remainder. In order to improve on this the number system is extended by introducing *rational numbersr*, which consist of all fractions $\dfrac{p}{q}$ where p and q are integers with $q \neq 0$.
Two fractions $\dfrac{p}{q}$ and $\dfrac{r}{s}$ define the same rational number if there is an integer $t \neq 0$ such that $r = tp$ and $s = tq$. Thus for example

$$\frac{1}{3} = \frac{2}{6} = \frac{-2}{-6} = \frac{3}{9} = \frac{-3}{-9} = \cdots .$$

Two operations exist for rational number: Addition and multiplication. They are defined as through new fractions

$$\frac{p}{q} + \frac{r}{s} = \frac{ps+qr}{qs} \qquad \text{og} \qquad \frac{p}{q} \cdot \frac{r}{s} = \frac{pr}{qs}.$$

In this manner we obtain exactly the same rules of arithmetic for rational number as we had for integers but in addition the rationals have the feature that any number $a \neq 0$ has a multiplicative invere which we denote by a^{-1}. If $a = \dfrac{p}{q}$ where $p \neq 0$ and $q \neq 0$ are integers, then $a^{-1} = \dfrac{q}{p}$. We also use the notation $\dfrac{1}{a}$ and $1/a$ for a^{-1}.
The rational number $\dfrac{0}{b}$ is called zero and it is also denoted by 0. Zero is the unit for addition, i.e. $a + 0 = a$ for all $a \in \mathbb{Q}$. The rational number $\dfrac{1}{1}$ is called one and it is also denoted by 1. It is a multiplicative unit, $1a = a$ for all $a \in \mathbb{Q}$.
We think of the integers as being a subset of the rational numbers $\mathbb{Z} \subset \mathbb{Q}$ by not distinguishing between the integer n and the rational $\dfrac{n}{1}$.
To have a mental image of the rational numbers, consider first the sets $\frac{1}{q}\mathbb{Z} = \{\frac{p}{q}; p \in \mathbb{Z}\}$ for each $q = 1, 2, 3, \ldots$. For $q = 1$ we have $\frac{1}{1}\mathbb{Z} = \mathbb{Z}$. On the other hand if $q > 1$, we draw $q - 1$ equally distributed points between any two numbers n and $n + 1$ in $\mathbb{Z}$ and these form our view of the rational numbers $n + k/q$, $k = 1, \ldots, q - 1$ which all lie between n and $n + 1$.



Example 2.3.1
Calculate

$$\frac{12}{5}\frac{16}{3}-\frac{5}{2}.$$

Solution: Here we have to be careful about order of operations and use rules for operations of rational numbers.

$$\frac{12}{5}\frac{16}{3}-\frac{5}{2}=\frac{12\cdot 16}{5\cdot 3}-\frac{5}{2}=\frac{192}{15}-\frac{5}{2}=\frac{384-75}{30}=\frac{309}{30}=\frac{103}{10}.$$

4 Powers

4.1 Powers

4.1.1 Handout

Powers and roots
Powers are introduced to simplify notation for repeated components. If a is a real number then we define $a^0 = 1$, $a^1 = a$, $a^2 = a\cdot a$, $a^3 = a\cdot a\cdot a$ and in general for a natural number we write $a^n = a\cdots a$ where there are a total of n terms in the product, all equal to a. For negative numbers n we define $a^n = 1/a^{-n}$.
The number a in the formula a^n is called the *base* and the number n is the *power*.
The following rules of arithmetic apply for powers:

$$\begin{aligned} a^n\cdot a^m &= a^{n+m},\\ \frac{a^n}{a^m} &= a^{n-m},\\ a^n\cdot b^n &= (ab)^n,\\ (a^n)^m &= a^{nm}. \end{aligned}$$

If $q\in\mathbb{N}$ and $a\in\mathbb{R}_+ = \{x\in\mathbb{R}; x\geq 0\}$, then there is exactly one number $x\geq 0$ such that $x^q = a$. This number x is called the q-th root of a and is denoted $\sqrt[q]{a}$. The following rules apply to roots:

$$\begin{aligned} \sqrt[q]{ab} &= \sqrt[q]{a}\cdot\sqrt[q]{b},\\ \sqrt[q]{\frac{a}{b}} &= \frac{\sqrt[q]{a}}{\sqrt[q]{b}},\\ \sqrt[q]{a^p} &= (\sqrt[q]{a})^p,\\ \sqrt[sq]{a^{sp}} &= \sqrt[q]{a^p},\\ \sqrt[sq]{a} &= \sqrt[s]{\sqrt[q]{a}}. \end{aligned}$$

In particular we write $\sqrt[2]{a}$ as $\sqrt{a}$ and we call this quantity the *square root* of a.
An obvious interpretation of the square root of a is that it gives the length of the edges line in a square with area a. The number $\sqrt[3]{a}$ is commonly called the *cube root* of a.
It is important to realise that roots do not obey addition. In general $\sqrt[q]{a+b}\neq\sqrt[q]{a}+\sqrt[q]{b}$. It is a very common error to assume these two sides are equal but taking a simple example

with e.g. $q = 2$ and $a = 9$, $b = 16$ and computing the value on each side of the formula shows that different values are obtained.
Now let r be a rational number and write this as a ratio of intergers $r = p/q$ wherem q is a natural number. Then define a to the power of r with the equation

$$a^r = \sqrt[q]{a^p}.$$

The fourth rule of roots above implies that the definition is independent of which fraction is chosen to represent the rational number r and the third rule implies that $a^r = (\sqrt[q]{a})^p$. By putting together rule for root and power computations one can show that the power ruled also apply for rational powers but in the general applications one much assume that the base a is positive or 0.

Pascal's triangle and the binomial theorem

Let p and q be natural numbers greater then one and $q < p$. Recall that $\binom{p}{q}$, the binomial coefficient $\frac{p!}{(p-q)!q!}$, is the number of ways to choose subsets with q elements from a set with p elements. Now consider a particular element a of the p elements from which the selection will be made. There are two possibilities considering a: Either a is in the selected subset or not. If a is in the subset then we must choose $q-1$ further elements of the remaining $p-1$ and this can be done in $\binom{p-1}{q-1}$ ways. On the other hand if a is not in the subset then we must choose all q elements for the subset from the $p-1$ which remain, which can be done in $\binom{p-1}{q}$ ways. We have therefore shown that $\binom{p}{q} = \binom{p-1}{q-1} + \binom{p-1}{q}$.
If we put p in place of $p-1$ in this formula we obtain $\binom{p+1}{q} = \binom{p}{q-1} + \binom{p}{q}$. Written in this form we can use the relationship to write the *Pascal-triangle*, which is an ordering of the binomial coefficients in a triangle where each element is the sum of the two closest elements in the line above.

$$\begin{array}{ccccccccc}
 & & & & \binom{0}{0} & & & & \\
 & & & \binom{1}{0} & & \binom{1}{1} & & & \\
 & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & \\
 & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & \\
\binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} \\
 & & & & \vdots & & & &
\end{array}
\quad = \quad
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & \underline{1} & & \underline{2} & & 1 & & \\
 & 1 & & \underline{3} & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1 \\
 & & & & \vdots & & & &
\end{array}$$

Here p is the line number under investigation and q is the number of the element within the line. Note that the triangle is symmetric about its center. This should not come as a surprise since by looking at the formula: $\binom{p}{p-q} = \frac{p!}{(p-(p-q))!(p-q)!} = \frac{p!}{q!(p-q)!} = \binom{p}{q}$. Also one notes that when we talk about selecting q from a collection of p, another person could write the statement in reverse and equivalently decide to select the $p-q$ which we did **not** select.
The p-th line in the Pascal-triangle gives the coefficients of $(a+b)^p$ as stated in the *binomial teorem*:

$$(a+b)^p = \sum_{q=0}^{p} \binom{p}{q} a^{p-q} b^q.$$

If i and j are such that $i + j = p$ then the coefficient of the term $a^i b^j$ becomes $\binom{i+j}{j} = \frac{(i+j)!}{i!j!}$. The Pascal-triangle therefore gives us

$$\begin{aligned}
(a+b)^2 &= a^2 + 2ab + b^2, \\
(a+b)^3 &= a^3 + 3a^2b + 3ab^2 + b^3, \\
(a+b)^4 &= a^4 + 4a^3b + 6a^2b^2 + 5ab^3 + b^4
\end{aligned}$$

and so forth.
Example 3.3.1
Let $a > 0$. Simplify $(a^{x+y})^z(a^{x-z})^y$.
Solution: We use rules of arithmetic for power and get:

$$(a^{x+y})^z(a^{x-z})^y = a^{zx+zy}a^{yx-zy} = a^{zx+zy+yx-zy} = a^{zx+yx} = (a^{z+y})^x.$$

Example 3.3.2
Expand $(x^2-1)^4$.
Lausn: We use the binomial theorem and get:

$$\begin{aligned}(x^2-1)^4 &= \binom{4}{4}(x^2)^4 + \binom{4}{3}(x^2)^3(-1) + \binom{4}{2}(x^2)^2(-1)^2 + \binom{4}{1}x^2(-1)^3 + \binom{4}{0}(-1)^4 \\ &= x^8 - 4x^6 + 6x^4 - 4x^2 + 1.\end{aligned}$$

Example 3.3.2
Let $a, b > 0$. Simplify

$$\sqrt[3]{\sqrt{a}^3 \sqrt[4]{b}^6}.$$

Solution: We use rules of arithmetic for roots and power:

$$\sqrt[3]{\sqrt{a}^3 \sqrt[4]{b}^6} = \sqrt[3]{\sqrt{a}^3}\sqrt[3]{\sqrt[4]{b}^6} = \sqrt{a}\sqrt[4]{b}^2 = \sqrt{a}\sqrt{b} = \sqrt{ab}.$$

5 Sets

5.1 Sets

5.1.1 Handout

Some basic concepts on sets
A set is a collection of separate objects or concepts. Two sets, A and B are said to be equal if they contain the same elements and we then write $A = B$.
The objects or concepts which define a set are its *elements*. We write $x \in A$ or $A \ni x$ if x is an element of the set A. If x is not an element of A then we write $x \notin A$ or $A \not\ni x$.
A set may be described by listing its elements. For example the set $\{1, 2, 3, \ldots\}$ is the set of all positive integers (natural numbers), $\{2, 4, 6, 8, 10\}$ is the set containing the first five even numbers and $\{2, 3, 5, 7, 11\}$ is the set containing the first five prime numbers.
A set is commonly described using an open expression $p(x)$, so that the set consists of all elements x such that $p(x)$ is a true expression. As an example, consider the even numbers

$$\{2, 4, 6, 8, \ldots\} = \{x \in \mathbb{N}; x \text{ is even}\}.$$

The expression here is "*x is an integer and x is even*".
The set A is said to be a *subset* of the set B is every element of A is also an element of B. This is denoted by $A \subset B$ or $A \subseteq B$. Note that the subsetsymbols $\subset$ and $\subseteq$ are equivalent and are used interchangably in mathematical texts.

The *empty set* is a set which contains no element. It is denoted by the symbol $\varnothing$. The empty set is considered to be a subset of all sets.

Operations on sets

If A and B are sets, then their *union*, $A \cup B$ is defined by

$$A \cup B = \{x; x \in A \text{ or } x \in B\},$$

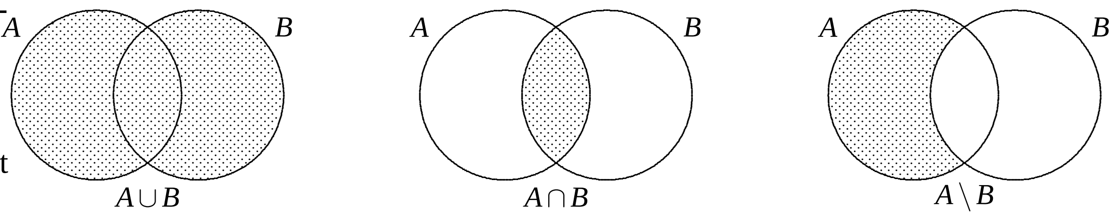
intersection, $A \cap B$ by

$$A \cap B = \{x; x \in A \text{ and } x \in B\}$$

and the *difference* between A and B is defined by

$$A \setminus B = \{x; x \in A \text{ and } x \notin B\}.$$

Note that the term "or" is used here, and always in mathematics is the meaning "and/or". When considering a subset A of a



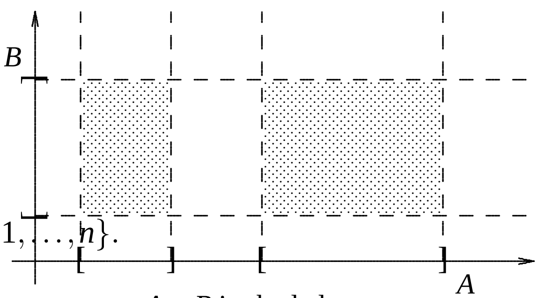
fixed set X, the set $X \setminus A$ is often termed the *complement* of the subset A and it is also denoted A^c or $\complement A$. The *product* or *product set*, $A \times B$, of two sets A and B is defined as the set of all pairs (a,b) of elements $a \in A$ and $b \in B$,

$$A \times B = \{(a,b); a \in A \text{ and } b \in B\}.$$

We also define unions and intersections of more than two sets. If $n \in \mathbb{N}$ and $A_1, A_2, \ldots, A_n$ are sets then we define their unions and intersections by

$$\bigcup_{i=1}^{n} A_i = \{x; x \in A_i \text{ for some } i = 1, \ldots, n\}$$

$$\text{and} \quad \bigcap_{i=1}^{n} A_i = \{x; x \in A_i \text{ for every } i = 1, \ldots, n\}.$$



$A \times B$ is shaded.

If we have an infinite sequence $A_1, A_2, A_3, \ldots$ of sets then we define their union and intersections by

$$\bigcup_{i=1}^{\infty} A_i = \{x; x \in A_i \text{ for some } i \in \mathbb{N}\} \quad \text{and} \quad \bigcap_{i=1}^{\infty} A_i = \{x; x \in A_i \text{ for every } i \in \mathbb{N}\}.$$

In the last two definitions the sets were enumerated. It is also possible to identify collections of sets using elements of sets which do not need to be subsets of the natural numbers. Such collections are denoted $(A_i)_{i \in I}$ where I (the index set) can be any set. We then denote the union and collection of all of these sets with

$$\bigcup_{i \in I} A_i = \{x; x \in A_i \text{ for some } i \in I\} \quad \text{and} \quad \bigcap_{i \in I} A_i = \{x; x \in A_i \text{ for every } i \in I\}.$$

Example 1.1.1
Given the sets $A := \{1,2,3,4,5\}$, $B := \{2,4,6,8,10\}$ and $C := \{6,7,8,9,10\}$.
a) Find $(A \cup B) \cap C$.
Solution: We start by finding $A \cup B$. That is the set of all members that are in at least one of the sets A, B, that is $A \cup B = \{1,2,3,4,5,6,8,10\}$.
$(A \cup B) \cap C$ has the members that belong to both $A \cup B$ and C. $(A \cup B) \cap C = \{6,8,10\}$.
b) Find $A \cup (B \cap C)$.
Solution: We can see that $B \cap C = \{6,8,10\}$ and then $A \cup (B \cap C) = \{1,2,3,4,5,6,8,10\}$
Notice that $A \cup (B \cap C) \neq (A \cup B) \cap C$. As you can see in this case the order of the operations matters as we are taking both union and intersection. Notice that $A \cap (B \cap C) = (A \cap B) \cap C$.
c) Find $(A \cap B) \cap C$.
Solution: We can see that $A \cap B = \{2,4\}$ and then $(A \cap B) \cap C = \{2,4\} \cap C = \emptyset$.

Example 1.1.2
Let $A \subset X$ and $B \subset Y$. Find the complement of $A \times B$ in $X \times Y$.
Solution: The complement contains all members $(a,b) \in X \times Y$ such that either $a \notin A$ or $b \notin B$ or both, which can be written as

$$(A \times B)^c = (A^c \times B) \cup (A \times B^c) \cup (A^c \times B^c).$$

Another way to write it is $(A^c \times Y) \cup (X \times B^c)$.

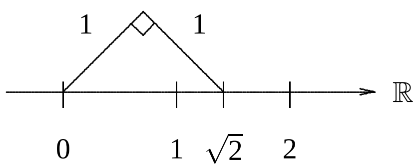
Example 1.1.3
a) Find $A \setminus B$ in terms of complements and intersections.
Solution: All members of A are either in B or B^c. The members of A that are not of B are thus exactly the members of $A \cap B^c$, so we have $A \setminus B = A \cap B^c$.
b) Let $p_1, p_2, \ldots, p_n$ be natural numbers. We denote by $p_i\mathbb{Z}$ the set of all integers divisible by p_i where $1 \leq i \leq n$. Denote the set of all numbers divisible by all of the numbers $p_1, p_2, \ldots, p_n$.
Solution: The set of numbers divisible by all of the numbers $p_1, \ldots, p_n$ are the intersection of numbers divisible by $p_1, p_2, \ldots p_n$, i.e.

$$\bigcap_{i=1}^{n} p_i\mathbb{Z}.$$

6 Number systems

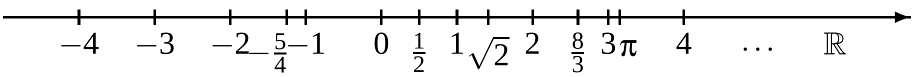
6.1 Number Systems

6.1.1 Handout



It is a natural question to ask whether each number on the number line corresponds to a rational number. This is equivalent to asking whether the length of any line segment can be given by a rational number. This was studied in detail by the ancient Greeks who eventually saw that the answer to the question is a negative. The rule of Pythagoras tells us that the length of the longest side, c, in a right-angled triangle with shorter sides both of length 1 must satisfy $c^2 = 2$. The resulting number is usually called the *square root of* 2 and is denoted by $\sqrt{2}$. It is not a rational number. This can be proved by an indirect proof, i.e. if we assume that the statement to be proved is incorrect, then we arrive at a contradiction.

The set of real numbers, $\mathbb{R}$, is designed to solve this conundrum. We view the rational numbers $\mathbb{Q}$ as points on the number line and enlarge the set of numbers so that there is a number a corresponding to every number on the line and let $\mathbb{R}$ be the set of all such numbers. As before we can view addition and multiplication as operations on the number line.



A real number which is not rational is called *irrational*. There is no special symbol for this set, which is $\mathbb{R} \setminus \mathbb{Q}$.
The usual rules of arithmetic apply for real numbers a, b and c:

$(a+b)+c = a+(b+c)$	*(associative rule for addition),*
$(ab)c = a(bc)$	*(associative rule for multiplication),*
$a+b = b+a$	*(commutative law for addition),*
$ab = ba$	*(commutative law for multiplication),*
$a(b+c) = ab+ac$	*(distributive law),*
$a+0 = a$	*(0 is the additive identity),*
$1a = a$	*(1 is the multiplicative identity).*

Every real number a has an additive inverse which is uniquely determined and we denote this by $-a$, and every real $a \neq 0$ has a multiplicative inverse a^{-1} which we commonly write as $\frac{1}{a}$, $1/a$ or $1 : a$.

Infinity

All of the number systems which have been discussed here, the natural numbers, integers, rational numbers and real numbers are *infinite and unlimited*, in one direction or both. In the upper direction this means that for any number a in each of these sets there is a number b in the same set such that $a < b$. When we want to use this property we need to put into use a new symbol, ∞, which we call the infinity symbol. We can think of this as a new symbol which is added to the set of real numbers in such a manner that $\infty > a$ for all a in $\mathbb{R}$ (and thus also for all a in $\mathbb{N}, \mathbb{Z}$ and $\mathbb{Q}$ since we view each of those as subsets of $\mathbb{R}$). In the same manner as we obtained the negative numbers from the positive numbers we define $-\infty$ so that $-\infty < a$ for all a in $\mathbb{R}$.
One must remember that ∞ and $-\infty$ are not numbers and should not be used as such. In particular we can not use them in ordinary arithmetic without special precautions although this can be tempting. As an example of the unusual nature of these symbols, we note that $-\infty$ does not have to be the additive inverse of ∞, i.e. the equation $-\infty + \infty = 0$ does not necessarily hold.

Ordering and intervals in $\mathbb{R}$

In $\mathbb{R}$ we have an ordering $<$ in such a manner that for any two numbers a and b one of three things must hold: $a < b$, $a = b$ or $b < a$. We also write $a > b$ if $b < a$. We have the following rules on the order of real numbers:

if $a < b$ and $b < c$, then $a < c$	*(ordering is transitive),*
if $a < b$ then $a+c < b+c$	*(ordering is unchanged under addition),*
if $a < b$ and $c > 0$, then $ac < bc$	*(ordering is unchanged under multiplication with a positive number),*
if $a < b$ and $c < 0$, then $bc < ac$	*(ordering is reversed under multiplication with a negative number).*

If $a, b \in \mathbb{R}$ and $a < b$, we define severaltypes of intervals:

$$\begin{array}{ll}
]a,b[=\{x\in\mathbb{R};\, a<x<b\} & \textit{(open interval),}\\
[a,b]=\{x\in\mathbb{R};\, a\le x\le b\} & \textit{(closed interval),}\\
[a,b[=\{x\in\mathbb{R};\, a\le x<b\} & \textit{(half-open interval),}\\
]a,b]=\{x\in\mathbb{R};\, a<x\le b\} & \textit{(half-open interval),}\\
]-\infty,a[=\{x\in\mathbb{R};\, x<a\} & \textit{(open halfline),}\\
]-\infty,a]=\{x\in\mathbb{R};\, x\le a\} & \textit{(closed halfline),}\\
]a,\infty[=\{x\in\mathbb{R};\, x>a\} & \textit{(open halfline),}\\
[a,\infty[=\{x\in\mathbb{R};\, x\ge a\} & \textit{(closed halfline),}\\
]-\infty,\infty[=\mathbb{R} & \textit{(the real line),}\\
[a,a]=\{a\} & \textit{(single point interval).}
\end{array}$$

Alternatively one may write (a,b) for $]a,b[$, $(a,b]$ for $]a,b]$ and so forth.
Each open interval contains infinitely many rational numbers and infinitely many irrational number.

$]a,b]$

On the real line the intervals are depicted using a thick line and the different types of endpoints are indicated by using a filled circle if the endpoint is within the interval but a clear circle if the endpoint is not a part of the interval.
For each $x\in\mathbb{R}$ we define the *absolute value* of x with

$$|x| = \begin{cases} x & x\ge 0,\\ -x & x<0. \end{cases}$$

The number $|x|$ measure the distance between 0 and x on the real line. For two real numbers x and y, the quantity $|x-y|$ measures the distance between them. If a and ε are real numbers and $\varepsilon>0$, then

$$\{x\in\mathbb{R};\, |x-a|<\varepsilon\} =]a-\varepsilon, a+\varepsilon[$$

is an open interval centered at a with diameter 2ε.

7 Countability

7.1 Countability

7.1.1 Handout

The number of elements of a set can be finite or infinite. The empty set $\varnothing$ contains no elements so the number of elements in it is 0. If A is a set and ther is $n\in\mathbb{N}$ and a bijective projection (one-to-one and onto function) $\{1,\dots,n\}\to A$, the set A is called *finite*. It is easily seen that the number n is uniquely determined, i.e. that one can not find two different numberd m and n in $\mathbb{N}$ and bijective projections $\{1,\dots,m\}\to A$ and $\{1,\dots,n\}\to A$. The number n is called the *number of elements of* A and often denoted by $\#A$.
If A is not a finite set we call A *infinite* and denote the number of elements by $\#A=\infty$ or $\#A=+\infty$. The set A is said to be *countably infinite* if there is a bijective projection $\mathbb{N}\to A$. One can also say that the set A is countably infinite if its elements can be placed into an infinite sequence $a_1,a_2,a_3,\dots$ where each element can appear exactly once.
The set of natural numbers $\mathbb{N}$ is of course countably infinite since the projection $x\mapsto x$ is bijective on $\mathbb{N}$. The set of integers $\mathbb{Z}$ is also countably infinite since we can set up a bijective projection $\mathbb{N}\to\mathbb{Z}$ as follows

$$1\mapsto 0,\ 2\mapsto 1,\ 3\mapsto -1,\ 4\mapsto 2,\ 5\mapsto -2,\dots.$$

Thus rule can be written more tightly as

$$n \mapsto \begin{cases} k, & \text{ef } n = 2k, \\ -k, & \text{ef } n = 2k+1. \end{cases}$$

The set of rational numbers $\mathbb{Q}$ is also countably infinite. Showing this is slightly more involved than proving that $\mathbb{Z}$ is countably infinite.

A set A is said to be *countable* if it is either finite or countably infinite and it is said to be *uncountable* if it is not countable. Several rules apply to countable sets, for example: *A set A is countable if and only if there is a one-to-one projection* $A \to \mathbb{N}$. Another rule is: *The countable union of countable sets is countable.* Formally, this states that sets of the form $A = \bigcup_{i \in I} A_i$ are countable if the index set I is countable and each of the sets A_i is countable.

The set of real numbers $\mathbb{R}$ is uncountable and the same can be said of the set of irrational numbers $\mathbb{R} \setminus \mathbb{Q}$.

Example 2.5.1

Tell whether the set $\mathbb{N} \times \mathbb{Z}$ is countable.

Solution: We have

$$\mathbb{N} \times \mathbb{Z} = \bigcup_{i=1}^{+\infty} \{i\} \times \mathbb{Z} = \bigcup_{i \in \mathbb{N}} \{i\} \times \mathbb{Z}$$

so $\mathbb{N} \times \mathbb{Z}$ is a countable union of countable sets and thus countable.

8 Algebraic expressions

8.1 Algebraic expressions

8.1.1 Handout

The word *algebra* has a wide meaning in mathematics. In general, algebra deals with the topic of defining and using rules of arithmetic on some set to investigate what the effects of the rules are. In this treatise we will investigate the effects of the rules of arithmetic for the real numbers which were under discussion earlier.

An expression or *formula*) is a collection of symbols, each of which can be numbers or alphabetical characters, connected together with the operators $+$, $-$, $\cdot$, :, fraction signs or parentheses. The alphabetical characters are viewed as symbols to denote real numbers These can be fixed numbers which have specific values or be variables which can take on any value within a subset of the real numbers.

As an example, consider

$$\frac{(3x(y+2)^2 - z)^3 - 2\frac{3}{4}(2x^2y+z)^4}{(ax^2 - 2xyz)^2 + 1.53}$$

This is a mixture of a variety of parentheses, power symbols, general fractions and decimal fractions. There are no limits to the complexity of this notation but rules must be observed on how the operations are used and the parentheses, ( ), must pair up. When computing the value of an expression, four rules of *operator precedence* must be followed:

1. *Anything inside a pair of parentheses must be evaluated without using anything outside the parentheses.*

2. *The addition (plus)* $+$*, and subtraction (minus)* $-$*, operators split the expressions into terms and each term must be evaluated completely before addition or subtraction is conducted.*

3. *The symbols for multiplicatioon, the dot $\cdot$ and cross $\times$, and for division, colon :, slash / and fraction symbol, split the above terms into components. They only apply to the component which immediately follows*

4. *Operations are conducted from left to right when terms are evaluated.*

Example 3.1.1
Simplify

$$\frac{\frac{14}{4}+27.5+(2^2+1)^2-12x(1-\frac{1}{3})}{(1-x)\frac{1}{2}+3}$$

Solution: We start by calculating in the parenthesis and taking together convenient components and get

$$\frac{\frac{14}{4}+27.5+(2^2+1)^2-12x(1-\frac{1}{3})}{(1-x)\frac{1}{2}+3}=\frac{\frac{7}{2}+27.5+(4+1)^2-12x\frac{2}{3}}{(1-x)\frac{1}{2}+3}=\frac{\frac{7+55}{2}+5^2-8x)}{\frac{1}{2}-\frac{x}{2}+3}.$$

We then extend by 2 and draw together similar components and cancel out:

$$=\frac{62+50-16x}{7-x}=\frac{112-16x}{7-x}=16\frac{7-x}{7-x}=16.$$

Example 3.1.2
Simplify

$$\left(2-\frac{5}{6}\right)^2+1\cdot\frac{5}{2}.$$

Solution: We start by calculating in the parenthesis and the right component. The rest is self explanatory.

$$\left(2-\frac{5}{6}\right)^2+1\cdot\frac{5}{2}=\left(\frac{7}{6}\right)^2+\frac{5}{2}=\frac{49}{36}+\frac{5}{2}=\frac{49+90}{36}=\frac{139}{36}.$$

9 Equations

9.1 Equations

9.1.1 Handout

Expansion and factorization
Expansion is the act of taking an expression consisting of a single term and changing this to many terms. The distributive law sets the stage for how this is done. A few resulting rules of expansion include the following:

$$\begin{aligned}
&(a+b)(c+d)=ac+ad+bc+bd,\\
&(a+b)^2=a^2+2ab+b^2 \qquad ,\\
&(a-b)^2=a^2-2ab+b^2 \qquad ,\\
&(a+b)(a-b)=a^2-b^2 \qquad ,\\
&(a+b)(a^2-ab+b^2)=a^3+b^3,\\
&(a-b)(a^2+ab+b^2)=a^3-b^3.
\end{aligned}$$

These expressions are *formulas* which state that the mathematical expressions on each side of the equal sign will yield the same number for any possible value of the variables a, b, c and d .

Consider in detail how the rules of arithmetic can be used to show that the first equation holds:

$$\begin{aligned}
(a+b)(c+d) &= (a+b)c+(a+b)d && \textit{(distributive law)},\\
&= c(a+b)+d(a+b) && \textit{(commutative law for multiplication)},\\
&= ca+cb+da+db && \textit{(distributive law)},\\
&= ac+bc+ad+bd && \textit{(commutative law for multiplication)},\\
&= ac+ad+bc+bd && \textit{(commutative law for addition)}.
\end{aligned}$$

Factorization is the inverse operation of expansion. In this case an expression with more than one term is converted into an equivalent expression which consists only of components which are multiplied together. One may think of factorization as applying expansion backards, as in $ab+ac=a(b+c)$, thus taking variables or entire components which appear in all terms and pulling them outside a parenthesis.

Sum and product symbols

Formulas involving long sums or products can be simplified by introdocing the symbols $\sum$ and $\prod$, the *sum-* and *product symbols*. If $a_1, a_2, \ldots, a_n$ are n mathematical symbols (where n might be ∞), then the sums and products of these symbols are denoted by

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + \ldots + a_n,$$

$$\prod_{i=1}^{n} a_i = a_1 \cdot a_2 \cdot \ldots \cdot a_n.$$

This notation is particularly convenient when there are simple ways to write the various a_i-values in a similar form.

Variations exist on these symbols, for exampls $\sum_{i=m}^{n} a_i$, where the sum starts at the m-th element and $\sum_{a\in A} a$, where the elements of A are added. Further, the symbol j or any other symbol can be used in place of i as an index.

The same comments apply to the multiplication symbol.

Example 3.2.1

Expand $(x-1)(x+1)^2$.

Solution:

$$(x-1)(x+1)^2 = (x-1)(x+1)(x+1) = (x^2-1)(x+1) = x(x^2-1)+(x^2-1) = x^3-x+x^2-1 = x^3+x^2-x-1.$$

Example 3.2.2

Expand $(x+4)^2(x-4)^2$.

Solution:

$$(x+4)^2(x-4)^2 = ((x+4)(x-4))^2 = (x^2-16)^2 = x^4-32x^2+256.$$

Example 3.2.3

Factorize x^3-2x^2+x.

Solution: We can see imminently that x divides the polynomial and after that we apply a common rule.

$$x^3 - 2x^2 + x = x(x^2 - 2x + 1) = x(x-1)^2.$$

Example 3.2.4
Factorize $x^4 - 1$.
Solution: We get

$$x^4 - 1 = (x^2+1)(x^2-1) = (x^2+1)(x+1)(x-1).$$

The polynomial x^2+1 is indivisible in the real numbers.

Example 3.2.5
Calculate the sum

$$\sum_{n=3}^{7} (n-1)^2$$

Solution: We start clarifying a bit by noticing that we can change the boundary of the sum to rewrite it and then calculate.

$$\sum_{n=3}^{7} (n-1)^2 = \sum_{2}^{6} n^2 = 4 + 9 + 16 + 25 + 36 = 90.$$

Example 3.2.6
Calculate

$$\prod_{n=1}^{5} 2n.$$

Solution: We just calculate:

$$\prod_{n=1}^{5} 2n = (2\cdot 1)\cdot(2\cdot 2)\cdot(2\cdot 3)\cdot(2\cdot 4)\cdot(2\cdot 5) = 2^5 \cdot 5! = 3840.$$

10 Inequalities and absolute values

11 Lines in the plane

11.1 Lines in the plane

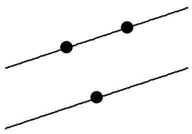
11.1.1 Handout

Points and lines in a plane

Points and lines are aspects of mathematics (or specifically of geometry) which correspond to everyday life. This is, however, why it is so difficult to decide where to start when discussing the field: We all *know* what lines and points are and therefore all attempts at formal definition appear clumsy and not needed at first sight.
The situation is not quite as simple as one might think. In order to conduct systematic investigations into geometry specific definitions of all concepts are needed and references to common knowledge are not adequate.

The methodology which has been selected to approach geometry is to define *points* and *lines* from some of their elementary properties, assume that some simple facts (so-called *axioms*) apply and work from there. The most important axioms on points and lines in classical geometry are:

- Any two distinct points define exactly one line which passes through both.
- A given line and point outside the line define exactly one line through the point and not intersecting the original line.



These axioms can be used to derive all of the geometry which applies to daily life.
Points in geometry have no size and lines extend infinitely in either direction without width. If A and B are points and ℓ is a line through the points then the part of the line between the points is called the *line seqment* or simply the *seqment* from A to B.
Normally no distinction is made between a line on the one hand and the set of points on the line on the other hand. Through any point A there are two half-lines parallel to a given line, ℓ, one in each direction. The point itself is on both halflines.

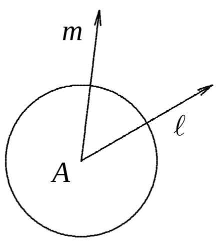
12 Triangles, other plane geometric figures and trigonometric function

12.1 Triangles

12.1.1 Handout

Angle

An *angle* is what is obtained when two halflines are drawn from the same initial point which is then called the vertex of the angle and the halflines are called its arms. The size of the angle is measured in *degrees* or *circular measure*. The size is quantified by first drawing a circle centered at the vertex and then computing the ratio between the length of the circle within the arms to the length of the full circle. If this ration is multiplied by 360 the result is the degrees of the angle. The circular measure of an angle is found by drawing a circle of length 1, centered in the vertex and measuring the length which falls between the arms. The measurement unit for the circular measure is rad, which is short for *radians*. Angles are measure with sign in such a way that the sign is positive if the angle is formed by turning a segment ℓ into a segment m counterclockwise across the angle and negative if the turn is clockwise.



The number π is defined as half of the length of a circle with radius 1, or its area. In decimals it is approximately $3.14159\ldots$. This is an irrational number and thus can not be represented by any repeated sequence of digits.
The area of a circle is proportional to its radius squared and the area of a circle with radius 1 is π. The area of a circle with general radius r is therefore πr^2 and its circumference is $2\pi r$.
The relationships between degrees and radians are simple. If an angle is x rad, then the size of the angle, measured in degrees is

$$y = \frac{x}{2\pi} \cdot 360^\circ = \frac{x}{\pi} \cdot 180^\circ.$$

Conversely, if the angle is y degrees then its size, measured in radians, is

$$x = \frac{y}{360} \cdot 2\pi \,\mathrm{rad} = \frac{y}{180} \cdot \pi \,\mathrm{rad}.$$

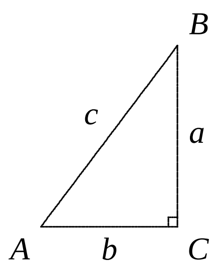
An angle is a *right angle* if it is a quarter of a circle, i.e if it is 90° or $\frac{\pi}{2}$ rad. An angle which is in absolute value less than en 90° is *sharp*, otherwise it is *obtuse*.
Example 4.2.1

a) Convert 75° to radians.
Solution: We multiply by $\frac{\pi}{180}$. The angle is $\frac{75}{180}\pi$ rad $= \frac{5}{12}\pi$ rad.
b) Convert $\frac{\pi}{6}$ rad to degrees.
Solution: We do this by multiplying by $\frac{180}{\pi}$. The angle is $\frac{\pi}{6}\frac{180}{\pi}^\circ = \frac{180}{6}^\circ = 30^\circ$.

Triangles, rectangles and polygons

A path which is composed of three line seqments joined together in pairs in three vertices is called a *triangle*. A path composed of four such line segments, joined at four vertices is a *quadrangle*, but a *pentagon* is the sides and vertices are five and in general a *n-sided polygon* if it has n lines and angles.
If all sides of an n-sided polygon are of the same length then it is *equilateral* and if all the angles are of the same size, all sides of the same length and none of them intersect, then it is *regular*.
In addition to these terms some concepts are specific for triangles and rectangles. Thus a triangle is a *right-angled triangle* if one of its angles is a right angle, and an *isosceles triangle* if any two sides are of equal length. A quadrangle is a *rectangle* if all angles are right angles and a regular quadrangle (thus also a rectangle) is a *square*.

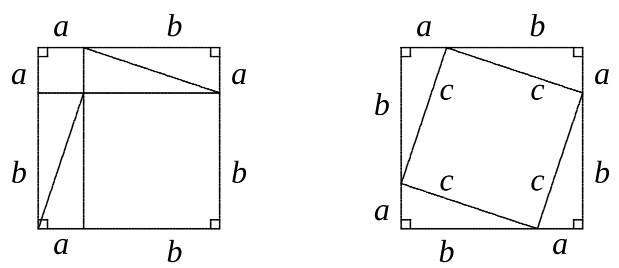
Pythagoras' theorem



Consider a triangle with vertices A, B and C and sides a, b and c as shown in the figure and that the angle C is a *right angle*, $C = 90^\circ$. Pythagoras' theorem tells us that the relationship between the lengths of the sides is given by the formula

$$a^2 + b^2 = c^2.$$

The theorem has a converse which states that if we have a triangle, ABC with opposite sides abc and the formula $a^2 + b^2 = c^2$ holds, then it is rectangular, i.e. $C = 90^\circ$.
Many methods can be used to prove Pythagoras' theorem. One of the most accessible is to consider the squares below. They both have side lengths $a+b$ and therefore the same area. If we look at the square on the left we see that it is composed of four right-angled triangles which each have area $\frac{1}{2}ab$ and two smaller squares with area a^2 on the one hand and b^2 on the other. The area of the square on the left is therefore $4 \cdot \frac{1}{2}ab + a^2 + b^2 = 2ab + a^2 + b^2$. Now consider the square on the right. It also consists of four right-angled triangles, each with area $\frac{1}{2}ab$ and a square with area c^2. The area of the square on the right is therefore $4 \cdot \frac{1}{2}ab + c^2 = 2ab + c^2$. As stated above, the two squares have the same area and therefore we have $2ab + a^2 + b^2 = 2ab + c^2$, which implies $a^2 + b^2 = c^2$.



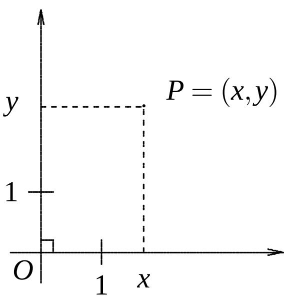
Example 4.4.1
Given a rectangular triangle with cathetus 4 og hypotenuse 5, find the other cathetus.
Solution: We use Pythagoras' theorem. That gives us the fallowing equation, where x denotes the unknown length, $5^2 = x^2 + 4^2$. That implies $25 - 16 = 9 = x^2$ and thus $x = \sqrt{9} = 3$.

Example 4.4.2
Given a rectangular triangle with hypotenuse 13 and cathetus 5, find the area of the triangle.
Solution: We need to find the product of the cathetuses, so we start by finding the unknown cathetus. It is $\sqrt{13^2 - 5^2} = \sqrt{144} = 12$. Then we can easily find the area: $A = 12 \cdot 5 \cdot \frac{1}{2} = 30$.

Example 4.4.3
Check if triangle with sides 5,6,7 is rectangular.
Solution: We use Pythagoras' theorem. The triangle is rectangular if and only if $7^2 = 6^2 + 5^2$, but $7^2 = 49$ and $5^2 + 6^2 = 61$, so it is not rectangular.

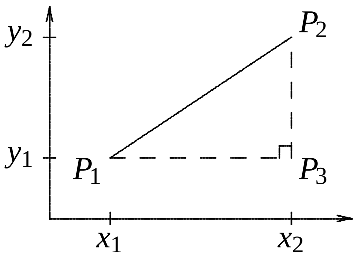
Coordinates and coordinate systems

The following is a short summary of some aspects of two-dimensional geometry, in specifically coordinate geometry. We start with some basic properties of a coordinate system in the plane.



Select a point, O, to be called the *origin* of the coordinate system, draw orthogonally two real number lines through it with the same scales and let the zero 0 on each line coincide with the origin O. Usually one line is chosen to run horizontally to the right and the other vertically upwards (this is not the only possible choice).
Now let P be a point in the plane. A vertical line through P will intersect with the horizontal axis in exactly one point which corresponds to a real number x called the *abscissa* of the point P. Similarly, a horizontal line through P intersects with the vertical axis in exactly one point which corresponds to a real number y which we call the *ordinate* of the point P. Put together we call the pair $(x,y) \in \mathbb{R} \times \mathbb{R} = \mathbb{R}^2$ the *coordinates* of the point P and we write $P = (x,y)$ to indicate the coordinates. Note that all points on a given vertical line have the same abscissa x and all points on a given horizontal line have the same ordinate y.

Distances between points



Now let $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ be two points in the plane and assume that they are not on a vertical line. Then consider the triangle with vertices P_1, P_2 and $P_3 = (x_2, y_1)$. This is a right-angled triangle since on the one hand P_1 and P_3 are on the same horizontal line with fixed ordinate y_1 and on the other hand P_2 and P_3 are on the vertical line with abscissa x_2. The distance between P_1 and P_3 is $|x_2 - x_1|$ and the distance between P_2 and P_3 is $|y_2 - y_1|$. If we denote the distance between P_1 and P_2 by $|P_1P_2|$, the Pythagoras' theorem implies

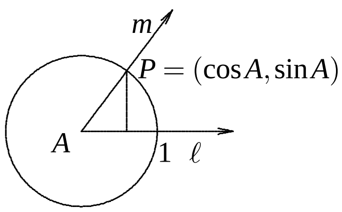
$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Example 4.5.1
What is the distance between $(1,2)$ and $(5,7)$?

Solution: We use the formula $\sqrt{(x-x_0)^2+(y-y_0)^2}$. Thus the distance is $\sqrt{(5-1)^2+(7-2)^2} = \sqrt{4^2+5^2} = \sqrt{41}$.

Example 4.5.2
A square with horizontal and vertical sides and middle in $(0,2)$ is drawn in a coordinate system and the distance from it's middle to the corners is 2.
Solution: We see that the squares' diagonal has length 4 so by Pythagoras it's side length is $2\sqrt{2}$. As the coordinates vary by a half side length from the middle each we can see the corners are in $(\sqrt{2},2-\sqrt{2}),(\sqrt{2},2+\sqrt{2}),(-\sqrt{2},2-\sqrt{2}),(-\sqrt{2},2+\sqrt{2}$.

Trigonometric functions



If A is the angle between the lines ℓ and m, then we define the cosine and sine of the angle A by displacing the image (or the coordinate system itself) so that A lands in the origin O of the coordinate system and then rotating the image around the origin O, so that ℓ ends up on the positive part of the horizontal axis. Next draw the unit circle centered at O and look at the intersection between the circle and the line m. This point has coordinates which we call $(\cos A, \sin A)$. This is our **definition** of the *cosine of A* and the *sine of A*.
If $\cos A \neq 0$, we also define the *tangent* of A by

$$\tan A = \frac{\sin A}{\cos A}$$

and if $\sin A \neq 0$, we define the *cotangent* of A by

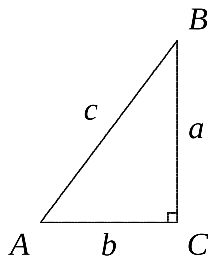
$$\cot A = \frac{\cos A}{\sin A}.$$

It is clear that the x-coordinate of a point on the unit circle is in the interval $[-1,1]$ and the same applies to its y-coordinate. An angle can be any real number and therefore the cosine and since are each functions with domain $\mathbb{R}$ and image $[-1,1]$. Note that the angles A and $A+2\pi$ give the sam intersection $P = (\cos A, \sin A)$. From this we see that both the cosine and the sine are periodic functions with period 2π.
Some values of the trigonometric functions can easily be derived, with the derivations left as exercises (these can be seen from considering properties of relevant triangles and judicious use of Pythagoras' theorem):

$^\circ$	rad	$\sin A$	$\cos A$	$\tan A$	$\cot A$
0	0	0	1	0	$--$
30	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$	$\sqrt{3}$
45	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	1
60	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{1}{\sqrt{3}}$
90	$\frac{\pi}{2}$	1	0	$--$	0

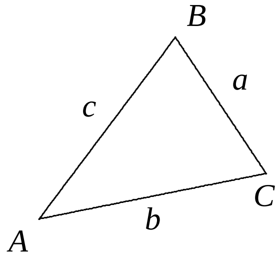
Similar triangles
Two triangles are *similar* if their angles are equal. This implies that the ration between corresponding sides is constant in the two triangles.
Trigonometric functions and triangles Consider the right-angled triangle in the figure. By setting up a similar triangle with longest side of length 1, the theorem on ratios between sides tells us that

$$\sin A = \frac{a}{c} = \text{opposite short side divided by long side,}$$
$$\cos A = \frac{b}{c} = \text{adjacent short side divided by long side,}$$
$$\tan A = \frac{a}{b} = \text{opposite short side divided by adjacent short side,}$$
$$\cot A = \frac{b}{a} = \text{adjacent short side divided by opposite short side.}$$



Areas

The area of a triangle is a half of the product of the baseline and the height. If we select an angle, say C, and consider a as the baseline, then the height is $b\sin C$. If we choose b as the baseline, then the height is $c\sin A$. The third option is to choose c as the baseline and then the height is $a\sin B$. The area, F, is therefore given by the following three formulae depending on the choice of baseline.



$$F = \tfrac{1}{2}ab\sin C = \tfrac{1}{2}bc\sin A = \tfrac{1}{2}ac\sin B.$$

Law of sines

The *Law of sines* states that the ration between the length of a side in a triangle and the sine of the opposite angle is the same for all three vertices and that this ration is $2R$, where R is the radius of the triangle's circumcircle:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R.$$

Law of cosines

The *Law of cosines* is a generalization of Pythagoras' theorem and gives the length on one side of a triangle when the others are known along with the angle between them:

$$c^2 = a^2 + b^2 - 2ab\cos C.$$

Example 4.6.1

A triangle ABC has side length $|AB| = 8, |AC| = 5$ and the corner $\angle BAC = 60^\circ$. What is the triangles area?

Solution: We use that $F = \frac{1}{2}bc\sin A$ where F is the area. Then we get

$$F = \frac{1}{2}|AB||AC|\sin(A) = \frac{1}{2}8\cdot 5\sin(60^\circ) = 10\sqrt{3}$$

Example 4.6.2

Let ABC and DEF be similar triangles with $\angle BAC = \angle EDF$, $\angle ABC = \angle DEF$, $\angle BCA = \angle EFD$. Let $|AB| = 10$, $|DE| = 5$, $|BC| = 6$. Find $|EF|$.

Solution: As the triangles are similar we know that

$$\frac{|AB|}{|BC|} = \frac{|DE|}{|EF|}$$

which rewrites

$$|EF| = \frac{|DE||BC|}{|AB|} = \frac{5\cdot 6}{10} = 3$$

Example 4.6.3
A triangle ABC has circumcircle with radius 5 and the angles $A = 60^\circ$ and $B = 45^\circ$. Find the side lengths of ABC.
Solution: Angle sum of triangle gives that $C = 75^\circ$. Then we can apply the law of sines which gives where a, b, c denote the sides of ABC that

$$\frac{a}{\sin(60^\circ)} = \frac{b}{\sin(45^\circ)} = \frac{c}{\sin(75^\circ)} = 2\cdot 5$$

and by solving the equations we get

$$a = 10\sin(60^\circ) = 5\sqrt{3}, \quad b = 10\sin(45^\circ) = 5\sqrt{2}, \quad c = 10\sin(75^\circ).$$

Example 4.6.4
An acute triangle ABC has side lengths $c = |AB| = 5,\ a = |BC| = 6,\ b = |CA| = 7.$ Find $\cos(\angle ABC)$.
Solution: We apply the law of cosine. We get $b^2 = c^2 + a^2 - 2ac\cos(\angle ABC)$. By isolating we get

$$\cos(\angle ABC) = \frac{c^2+a^2-b^2}{2ac} = \frac{5^2+6^2-7^2}{2\cdot 5\cdot 6} = \frac{12}{60} = \frac{1}{5}.$$

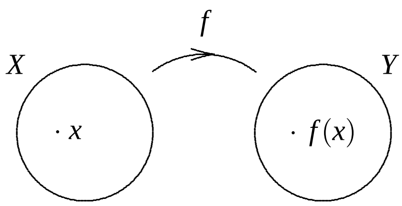
13 Circles

14 Functions

14.1 Functions

14.1.1 Handout

Functions and graphs
A *function* or *projection* f from a set X into a set Y is a rule, which assigns to each element $x \in X$ exactly one element $f(x)$ in Y.



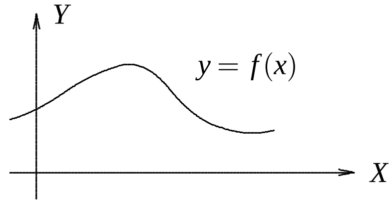
Functions are written in several ways, for example

$$f : X \to Y, \qquad X \xrightarrow{f} Y \qquad \text{or} \qquad X \to Y,\ x \mapsto f(x).$$

The set X is the *domain* of the function f and the set Y is the *target set* of the function. In this case f is *defined* on the set X and that it *takes on values* in the set Y. If $x \in X$, then $f(x)$ is the *value of the function f at x*. The set of all values of the function f is the subset $\{f(x); x \in X\}$ of Y and it is termed the *image* of the function f.

The *graph* of the function f is a subset of the product set $X\times Y$ defined by the collection of pairs

$$\{(x,y)\in X\times Y\,;\,y=f(x)\}.$$



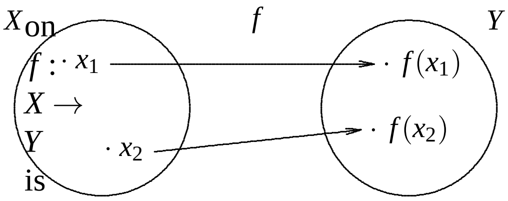
$\{(x,y)\in X\times Y\,;\,y=f(x)\}$

If $A\subset X$, then the set $\{f(x)\,;\,x\in A\}$ is the *image* of f, denoted by $f(A)$.
If $B\subset Y$, then the set of all $x\in X$ such that $f(x)\in B$ is the *inverse image* of the set B under the function f. This set is denoted by $f^{-1}(B)$,

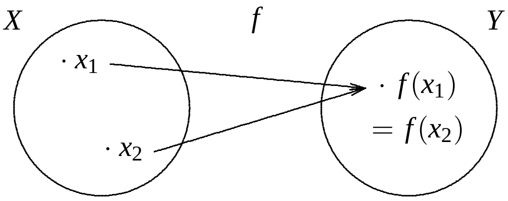
$$f^{-1}(B)=\{x\in X\,;\,f(x)\in B\}.$$

If it is clear what function is under consideration, then $f^{-1}(B)$ is call the inverse image of B.

One-to-one and onto functions
A function $f:X\to Y$ is said to be *one-to-one* (1-1 or *injective* if $x_1\neq x_2$ implies $f(x_1)\neq f(x_2)$. Equivalently, $f(x_1)=f(x_2)$ implies $x_1=x_2$. In words, the function maps different points in the original set (domain) to different points in the image.



f is 1-1



f is not 1-1

A function $f:X\to Y$ is said to be *onto* (or *surjective*) if for every $y\in Y$ there is an $x\in X$ such that $f(x)=y$.
A function $f:X\to Y$ is said to be *one-to-one and onto* or *bijective* if it is both 1-1 and onto.
These concepts can usefully be considered from solutions to equations. Take y to be an element of Y and we want to see whether we can find $x\in X$ which solves the equation

$$f(x)=y.$$

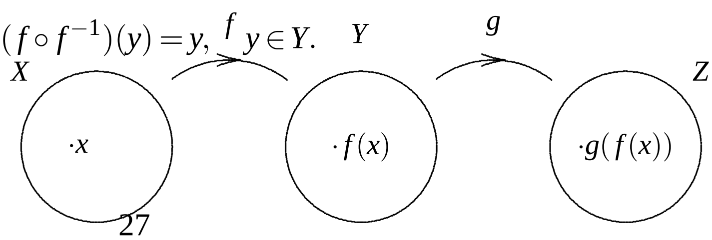
The function f is onto if and only if a solution x can be found of every $y\in Y$. The function f is 1-1 if and only if every solution is unique. Hence the function f is bijective if and only if the equation has exactly one solution x for each $y\in Y$.
If $f:X\to Y$ is bijective, then we can define another function $f^{-1}:Y\to X$ by defining $f^{-1}(y)=x$ where x is the unique solution to the equation $f(x)=y$. We call f^{-1} the *inverse* of the function f.

Composition of functions If $f:X\to Y$ and $g:Y\to Z$ are two functions then we define their composition $g\circ f:X\to Z$ by the formula

$$(g\circ f)(x)=g(f(x)).$$

For bijective functions f we have

$(f^{-1}\circ f)(x)=x,\quad x\in X$ and $(f\circ f^{-1})(y)=y,\quad y\in Y.$



Example 1.2.1
What is the inverse of the function $f : \mathbb{R}_+ \to \mathbb{R}_+, f(x) := \frac{1}{x}$?
(Here $\mathbb{R}_+ =]0, +\infty[$).
Solution: We want to find a function $f^{-1} : \mathbb{R}_+ \to \mathbb{R}_+$ such that $f^{-1}(\frac{1}{x}) = x$, but that function is f, that is $f^{-1} = f$.

Example 1.2.2
Given the functions $f : \mathbb{R}_+ \to \mathbb{R}_+$; $f(x) := \frac{1}{x}$ and $g : \mathbb{R}_+ \to \mathbb{R}$; $g(x) := \ln(x)$, tell whether $g \circ f$ is onto, one-to-one or bijective.
Solution: As we saw in last example f is bijective. The natural logarithm is strictly increasing so it is one-to-one and it can take as high or low values as one wants so it is onto and thus bijective. Then $g \circ f$ is composition of bijective functions and therefore bijective.

Operations on functions
A function f is *real-valued* or a *real function* if all of its outcomes, $f(x)$ are in $\mathbb{R}$. The function f is *complex valued* or a *complex function* if its outcomes are in $\mathbb{C}$. The number $f(x)$ is called the *value of the function f at x*.
If f and g are two functions defined on a set X, then we can use the operations on $\mathbb{R}$ to define new functions. For every two functions f ans g on the set X we define three new functions which we call the *sum*, *difference* and *product* of the functions, denoted by $f+g$, $f-g$ and fg (or $f \cdot g$). These are the functions defined by

$$\begin{aligned} (f+g)(x) &= f(x)+g(x), \qquad x \in X, \\ (f-g)(x) &= f(x)-g(x), \qquad x \in X \\ (fg)(x) &= f(x)g(x), \qquad x \in X. \end{aligned}$$

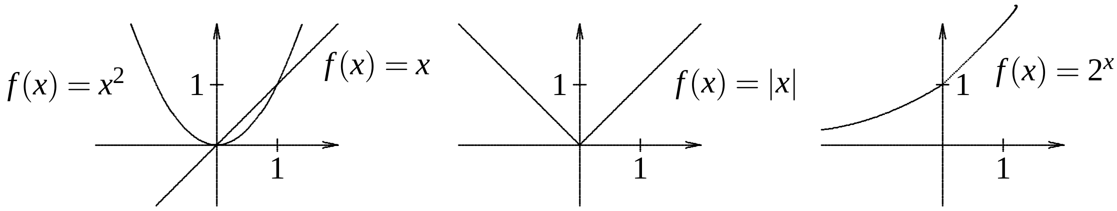
If $g(x) \neq 0$ for all $x \in X$, then we can also define the *ratio* $\dfrac{f}{g}$ of the functions f and g with the formula

$$\frac{f}{g}(x) = \frac{f(x)}{g(x)}.$$

The ratio can also be denoted by f/g or $f : g$. If the function g has zeros, then the ratio can be defined in the same manner but the resulting function has domain $X \setminus \{x \in X \,;\, g(x) = 0\}$.
Mathematical analysis largely revolves around defining functions and investigating their behaviour. Consider some particular properties of functions:
A periodic function is a function for which there exists a number a such that $f(x) = f(x+a)$ for all x in the domain of f. The number a is the *period* of the function f.
A constant function can be defined on any domain but only takes on a single value c i.e. is of the form $x \mapsto c$ where c is a given constant.
A polynomial is a function $\mathbb{R} \to \mathbb{R}$ of the form

$$x \mapsto a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where $n \geq 0$ is an natural number and $a_0, \ldots, a_n$ are real numbers.
A power function is of the form $f(x) = x^n$ where n is a real number. If $n \geq 0$ is an integer, then the domain can be $\mathbb{R}$.
The absolute value function is the function $\mathbb{R} \to \mathbb{R}$, $x \mapsto |x|$.
An exponential function is a function $\mathbb{R} \to \mathbb{R}$ of the form $x \mapsto a^x$ where $a > 0$ is a real number.



Power functions Absolute value function Exponential function

Example 2.6.1
Given the functions $f:[0,5]\to\mathbb{R};\ f(x):=x^2+2x-3$ and $g:[0,5]\to\mathbb{R};\ g(x):=x+3$. Denote the function $\frac{f}{g}$.
Solution: g has no zeros in it's domain. Now $x^2+2x-3=(x-1)(x+3)$. We get

$$\frac{f}{g}:[0,5]\to\mathbb{R};\ \frac{f}{g}(x)=\frac{f(x)}{g(x)}=\frac{(x-1)(x+3)}{x+3}=x-1.$$

15 Polynomials

15.1 Polynomials

15.1.1 Handout

A polynomial is a formula of the form

$$p(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$$

where the coefficients a_j are real numbers. The *degree* of the polynomial is the largest j such that $a_j\neq 0$. Solutions to the equation $p(x)=0$ are called the *roots* or *zeroes* of the polynomial p.
Solving the linear equation
A first degree polynomial is of the form $ax+b$, where $a\neq 0$ and b are some real numbers and an equation of the form $ax+b=c$ could be called a first degree equation though it is more commonly called a linear equation. This equation is usually standardized by subtracting c from both sides and then reeplacing $b-c$ by b, so it becomes $ax+b=0$. This equation has exactly one solution, namely $x=-b/a$.
Solving the quadratic equation
A second degree polynomial is of the form ax^2+bx+c where $a\neq 0$, b and c are real numbers. A second degree equation is of the general form $ax^2+bx+c=d$, where d is a real number but d is normally subtracted from both sides of the equation to obtain the standardized form of the quadratic equation, $ax^2+bx+c=0$.
Now let us solve the equation $ax^2+bx+c=0$. First simplify the task by dividing a into both sides and obtain an equivalent equatioon $x^2+Bx+C=0$, where we have defined $B=b/a$ and $C=c/a$. The second step is to look closer at the first two terms here, x^2+Bx and think about how this can be written as a squared term plus a constant. In other words we want to find α and rewrite the two terms as $(x+\alpha)^2$ plus a constant. By expanding this square we know that $(x+\alpha)^2=x^2+2\alpha x+\alpha^2$. Hence we see that we need to have $2\alpha=B$ to get

$$0=x^2+Bx+C=(x+\frac{B}{2})^2-\frac{B^2}{4}+C,$$

where we have basically added the final term and subtracted it again to make the equations hold. From this we see that the original equation is equivalent to

$$0 = (ax^2+bx+c)/a = \left(x+\frac{b}{2a}\right)^2 - \frac{b^2}{4a^2}+\frac{c}{a}.$$

By subtracting $-b^2/(4a^2)+c/a$ from both sides we again obtain an equivalent equation

$$\left(x+\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2}-\frac{c}{a} = \frac{b^2-4ac}{4a^2}.$$

The quantity $D = b^2-4ac$ is the *discriminant* of the equation, since the sign of this number defines how many solutions there are to the equation. If $D > 0$, there are two solutions

$$x_1 = \frac{-b+\sqrt{D}}{2a} \qquad \text{og} \qquad x_2 = \frac{-b-\sqrt{D}}{2a}.$$

If $D = 0$, there is a single solution

$$x = \frac{-b}{2a}.$$

If $D < 0$, there is no solution among the real numbers since a real number squared is always positive.

Polynomial division

If p and q are polynomials then one can sometimes fing a polynomial k such that $p = kq$. In this case we say that the polynomial q *divides* the polynomial p and the operation of finding the k in question is called *polynomial division*.

As in division with a remainder for integers one can for polynomials p and q find polynomials k and r where r is of a lower degree than q such that $p(x) = k(x)q(x)+r(x)$.

Factoring polynomials

Let p be a polynomial, a be a number and$q(x) = x-a$. Then there are polynomials $k(x)$ and $r(x)$ such that $p(x) = (x-a)k(x)+r(x)$ where $r(x)$ is of a degree less that one and hence $r(x)$ is a constant, possibly 0. By inserting a for x we can obtain the value of this constant. We then see that $r(x) = p(a)$, and if a is a root of the polynomial p, i.e. $p(a) = 0$, then this implies that $x-a$ divides $p(x)$, i.e. $x-a$ is a factor in $p(x)$.

From this we see that $x-a$ divides $p(x)$ if and only if a is a root of $p(x)$.

Example 3.4.1

Solve the equation $5x-7 = 2x$.

We solve a first degree equation by isolating x. We get $3x = 7$ which is equivalent to $x = \frac{7}{3}$.

Example 3.4.2

Solve the equation $2x^2+3x-5 = 2$.

Solution: We want the equation on standard form, that is $2x^2+3x-7 = 0$. Then we use the solution formula for quadratic equation.

$$x = \frac{-3\pm\sqrt{3^2-4\cdot 2\cdot(-7)}}{2\cdot 2} = \frac{-3\pm\sqrt{65}}{4}.$$

The solutions are $\frac{-3-\sqrt{65}}{4}$ and $\frac{-3+\sqrt{65}}{4}$.

Example 3.4.3

Divide x^2+2x-4 by $x+4$ with remainder.

Solution: We want the rest to be constant so we choose a polynomial $ax+b$ such that

$(ax+b)(x+4) = ax^2+4ax+bx+4b = x^2+2x+c$ where the constant c can be any number. Then $a=1$ and $4a+b=4+b=2$ so $b=-2$, then $c=4b=-8$. Then we have the residue 4.

$$x^2+2x-4 = (x+4)(x-2)+4.$$

In traditional form we write

$$\begin{array}{r|r} & x-2. \\ \hline x+4 & x^2+2x-4 \\ & -x^2-4x \quad \\ \hline & -2x-4 \\ & 2x+8 \\ \hline & 4 \end{array}$$

16 Rational functions

17 Exponential functions

18 Inverses of functions

19 Logarithms

20 Trigonometric functions and the unit circle

20.1 Trigonometric functions

20.1.1 Handout

The trigonometric functions satisfy many equations which we commonly need in computations. Some of these will be derived in the following.

A *reflection about the horizontal axis* is the projection $\mathbb{R}^2 \to \mathbb{R}^2$ given by the formula $(x,y) \mapsto (x,-y)$. Consider the vector (x,y) in polar coordinates, $(x,y) = (r\cos\theta, r\sin\theta)$. The reflection has length r and angular coordinate $-\theta$. The description of the reflection in polar coordinates is therefore $(x,y) = (r\cos\theta, r\sin\theta) \mapsto (r\cos(-\theta), r\sin(-\theta))$. If we connect these two descriptions together we obtain the formulae

$$\cos(-\theta) = \cos\theta \qquad \text{og} \qquad \sin(-\theta) = -\sin\theta.$$

A *reflection about the vertical axis* is the projection $\mathbb{R}^2 \to \mathbb{R}^2$ given by the formula $(x,y) \mapsto (-x,y) = (-r\cos\theta, r\sin\theta)$. From the image we see that the reflection has the angular coordinate $\pi-\theta$ so it can be written as $(r\cos\theta, r\sin\theta) \mapsto (r\cos(\pi-\theta), r\sin(\pi-\theta))$. From this we conclude that

$$\cos(\pi-\theta) = -\cos\theta \qquad \text{and} \qquad \sin(\pi-\theta) = \sin\theta.$$

A *reflection about the origin* is the function $\mathbb{R}^2 \to \mathbb{R}^2$ given by $(x,y) \mapsto (-x,-y) = (-r\cos\theta, -r\sin\theta)$. Now, this reflection can be viewed as a rotation of magnitude π around the origin so this function can also be described by the formula $(r\cos\theta, r\sin\theta) \mapsto (r\cos(\theta+\pi), r\sin(\theta+\pi))$. We have thus shown that

$$\cos(\theta+\pi) = -\cos\theta \qquad \text{and} \qquad \sin(\theta+\pi) = -\sin\theta.$$

A *reflection about the line* $y = x$ is the projection $\mathbb{R}^2 \to \mathbb{R}^2$ given by $(x,y) \mapsto (y,x) = (r\sin\theta, r\cos\theta)$. The angular coordinate of the image is $\frac{1}{2}\pi - \theta$ and hence the function can also be described by $(r\cos(\frac{1}{2}\pi-\theta), r\sin(\frac{1}{2}\pi-\theta))$. This leads to the formula

$$\cos(\tfrac{1}{2}\pi-\theta) = \sin\theta \qquad \text{and} \qquad \sin(\tfrac{1}{2}\pi-\theta) = \cos\theta.$$

Now consider two vectors $\mathbf{a} = (a_1, a_2)$ and $\mathbf{b} = (b_1, b_2)$ of length 1 and write them using polar coordinates $\mathbf{a} = (\cos\theta, \sin\theta)$ and $\mathbf{b} = (\cos\varphi, \sin\varphi)$. The angle between the vectors is $\theta-\varphi$, so $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos(\theta-\varphi) = a_1b_1 + a_2b_2$ is in this case

$$\cos(\theta-\varphi) = \cos\theta\cos\varphi + \sin\theta\sin\varphi.$$

If we now interchange φ and $-\varphi$ and use the formulae $\cos(-\varphi) = \cos\varphi$ and $\sin(-\varphi) = -\sin\varphi$ which were derived earlier we obtain

$$\cos(\theta+\varphi) = \cos\theta\cos\varphi - \sin\theta\sin\varphi.$$

This gives

$$\begin{aligned}\sin(\theta+\varphi) &= \cos(\tfrac{1}{2}\pi-\theta-\varphi) = \cos((\tfrac{1}{2}\pi-\theta)-\varphi)\\ &= \cos(\tfrac{1}{2}\pi-\theta)\cos\varphi + \sin(\tfrac{1}{2}\pi-\theta)\sin\varphi\\ &= \sin\theta\cos\varphi + \cos\theta\sin\varphi.\end{aligned}$$

If we now interchange φ and $-\varphi$ in this formula and use tha fact that $\cos(-\varphi) = \cos\varphi$ and $\sin(-\varphi) = -\sin\varphi$, then we obtain

$$\sin(\theta-\varphi) = \sin\theta\cos\varphi - \cos\theta\sin\varphi.$$

We can summarize these results in the following table:

$$\begin{aligned}
\cos(-\theta) = \cos\theta \quad &\text{og} \quad \sin(-\theta) = -\sin\theta,\\
\cos(\pi-\theta) = -\cos\theta \quad &\text{and} \quad \sin(\pi-\theta) = \sin\theta,\\
\cos(\theta+\pi) = -\cos\theta \quad &\text{and} \quad \sin(\theta+\pi) = -\sin\theta,\\
\cos(\tfrac{1}{2}\pi-\theta) = \sin\theta \quad &\text{and} \quad \sin(\tfrac{1}{2}\pi-\theta) = \cos\theta,\\
\cos(\theta-\varphi) &= \cos\theta\cos\varphi + \sin\theta\sin\varphi,\\
\cos(\theta+\varphi) &= \cos\theta\cos\varphi - \sin\theta\sin\varphi,\\
\sin(\theta+\varphi) &= \sin\theta\cos\varphi + \cos\theta\sin\varphi,\\
\sin(\theta-\varphi) &= \sin\theta\cos\varphi - \cos\theta\sin\varphi.
\end{aligned}$$

The last four formulae are the *angle sum and difference identities*. Consider the formula for $\cos(\theta-\varphi)$ in the special case when $\varphi = \theta$. Then $\cos(\theta-\varphi) = \cos 0 = 1$ and we obtain

$$\cos^2\theta + \sin^2\theta = 1.$$

This formula simply states that the point $(\cos\theta, \sin\theta)$ is on the unit circle. If we stick to the case $\varphi = \theta$ and consider the formula for $\cos(\theta+\varphi)$, we obtain

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1.$$

Similarly we obtain

$$\sin(2\theta) = 2\sin\theta\cos\theta.$$

Example 5.5.1
Simplify

$$\frac{\sin(2x)}{\sin^2(x)}\tan(x)$$

Solution: We start by using double angle rule for sine and write tan as function of sin and cos. Then we get:

$$\frac{\sin(2x)}{\sin^2(x)}\tan(x) = \frac{2\sin(x)\cos(x)}{\sin^2(x)}\frac{\sin(x)}{\cos(x)} = \frac{2\sin^2(x)\cos(x)}{\sin^2(x)\cos(x)} = 2.$$

Example 5.5.2
Simplify

$$(\cos(2y)+\sin^2(y))\frac{\sin(x-y)+\cos(x)\sin(y)}{\sin(x)\cos^2(y)}.$$

Solution: We use double-angle rule for cos and angle-sum rule for sin and get

$$\begin{aligned}
&(\cos(2y)+\sin^2(y))\frac{\sin(x-y)+\cos(x)\sin(y)}{\sin(x)\cos^2(y)}\\
&= (\cos^2(y)-\sin^2(y)+\sin^2(y))\frac{\sin(x)\cos(y)-\cos(x)\sin(y)+\cos(x)\sin(y)}{\sin(x)\cos^2(y)}\\
&= \cos^2(y)\frac{\sin(x)\cos(y)}{\sin(x)\cos^2(y)}\\
&= \cos(y).
\end{aligned}$$

Example 5.5.3
Calculate exact value of $\sin(105^\circ)$.
Solution: We use the angle-sum rule for sine:

$$\sin(105^\circ) = \sin((135-30)^\circ) = \sin(135^\circ)\cos(30^\circ)-\cos(135^\circ)\sin(30^\circ) = \frac{1}{\sqrt{2}}\frac{\sqrt{3}}{2} - \frac{-1}{\sqrt{2}}\frac{1}{2} = \frac{\sqrt{3}+1}{\sqrt{8}}.$$

21 Theorems and proofs

22 Combinatorics

22.1 Combinatorics

22.1.1 Handout

Counting and the !-symbol
Assume that some form of selection is performed in n steps where there are k_i possibilities of selection in the ith step. In other words, in the first step we select one out of k_1 possibilities, in the next one of k_2 possibilities and so forth. Using induction it is easy to see that the total number of different outcomes is equal to

$$k_1 \cdot k_2 \cdot \ldots \cdot k_n.$$

An example will clarify this multiplicative principle: A person plans to dine at a restaurant which offers three different appetizers, four main courses and two types of dessert. The principle states that the person can choose a three course meal in exactly $3 \cdot 4 \cdot 2 = 24$ different ways.

Choosing several objects from the same set often results in sequences of factors of the form $p \cdot (p-1) \cdot (p-2) \cdot \ldots \cdot (p-q)$ for some numbers p and q. To simply working with such products we define for a natural number p the quantity

$$p! = p \cdot (p-1) \cdot (p-2) \cdot \ldots \cdot 2 \cdot 1$$

pronounced p *factorial*. For convenience this operation is also defined for 0 by setting $0! = 1$.

Imagine a set with p different elements and we want to investigate in how many ways these elements can be put into an ordered sequence. The first element of the sequence can be chosen in p ways, the next in $p-1$ ways etc. The total number of orderings is therefore $p \cdot (p-1) \cdot (p-2) \cdot \ldots \cdot 2 \cdot 1 = p!$.

Permutations and combinations

Sometimes elements need to be selected out of a set and placed in order. Consider selecting q elements out of a total of p elements in the set A. Such a choice of elements is termed a *q-permutation* from A. The first element can be selected in p ways, the next in $p-1$ ways etc., until we come to the last element which can be selected in $p-q+1$ different ways. The rules of counting above imply that this can be done in $p \cdot (p-1) \cdot \ldots \cdot (p-q+1) = \frac{p!}{(p-q)!}$ ways in total. The number of q-permutations of p elements is sometimes denoted $(p)_q$ and we have shown that $(p)_q = \frac{p!}{(p-q)!}$.

Now assume that the order of the selected elements is irrelevant so we are only interested in knowing how many different combinations of elements can be selected. Since the above computations count the different ordering separately, even if they contain the same elements, they result in higher counts than we now want. Suppose that the q elements have been selected without regards to order. Then there are a total of $q!$ ways to place them in order. It follows by counting the number of ordered sequences we have over-counted by a factor of $q!$ when we want the unordered counts. Therefore the number of ways in which q elements can be selected from a set of p without regards to order is $\frac{p!}{(p-q)!q!}$, i.e. $\frac{1}{q!}(p)_q$, which is $1/q!$ times the number of q-orderings from p.

A selection of this form of q elements where order is irrelevant is called a *q-combination*. The number of q-combinations from p elements is denoted $\binom{p}{q}$ and called the *binomial coefficient*.

Consider forming a team of 7 handball players from a group of 10 players. If we start just by selecting team members then this can be done in $\binom{10}{7} = \frac{10!}{(10-7)!7!}$ ways as the number of 7-combinations from 10 elements. Following this the players have to be placed in the different positions. That can be done in 7! ways. In the end the total number of possible different teams is $7!\binom{10}{7} = 7!\frac{10}{(10-7)!7!} = \frac{10}{(10-7)!} = (10)_7$, i.e. equal to the total number of 7-permutations of 10 elements.

Example 2.1.2

a) After a dance a driver asks a group of people if they need a ride. 10 people need a ride but there is only room for four, in how many ways can the group of passengers be given that all seats are filled?

Solution: We choose four from a group of 10, but the number of such combinations is

$$\binom{10}{4} = \frac{10!}{6!4!} = 7 \cdot 3 \cdot 10 = 210.$$

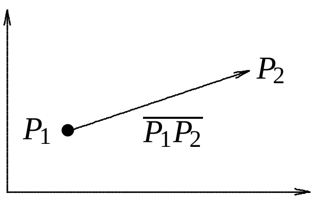
b) As the passengers have been chosen it is still left to choose who is in the front seat. If we do not take into account different orders in the back seat, in how many ways can the driver pick passengers to his car?
Solution: We can choose the group of passengers in 210 ways and then we have four options for the front seat, so we get $4 \cdot 210 = 840$ possibilities.

23 Vectors

23.1 Vectors

23.1.1 Handout

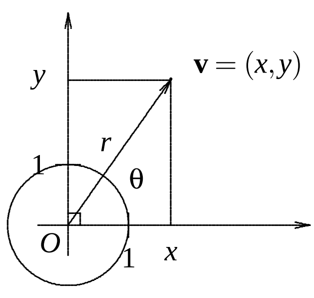
Vectors



Let $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ be two points in a plane and draw a line segment from P_1 to P_2 and orientate it from P_1 to P_2. This directed quantity is called the *vector from* P_1 *to* P_2 and is denoted by $\overline{P_1P_2}$ and in a figure it is drawn as an arrow from P_1 to P_2. The number $x_2 - x_1$ is the *ordinate* or the *x-coordinate* of the vector and the number $y_2 - y_1$ is the *abscissa* or *y-coordinate*. The coordinate of the vector is the tuple $(x_2 - x_1, y_2 - y_1)$. In some texts the coordinates of vectors are distinguished from the coordinates of points by using either different parentheses as in $[a, b]$ or as columns $\begin{bmatrix} a \\ b \end{bmatrix}$. Usually this is not of great importance since it is usually clear from the context whether the reference is to a vector or point.
Now think of another set of points, P_3 and P_4. If the directed line seqment from P_3 to P_4 is such that when P_3 is translated to P_1, then P_4 is translated to P_2, we say that P_1P_2 and P_3P_4 define the same *vector*. In this case the coordinates of $\overline{P_1P_2}$ are the same as the coordinates of $\overline{P_3P_4}$.
A vector is therefore a directed line segment without a fixed starting or endpoint. If it is drawn in the plane as a vector from one point to another then its coordinates are the same, no matter where it is drawn. It is customary to indicate vectors in boldface in print, cf. **a**, **b**, **c**... or on a whiteboard with an underline as in *a*, *b*, *c*.... The zero vector **0** is the vector with coordinates $(0, 0)$.
Each point P in the plane defines the vector $\overline{OP}$ from the origin O to P. This is the *position vector* of the point P. It coordinates are the same as the coordinates of the point P. In this manner we obtain a bijective projection from the set of all points in the plane and the set of all vectors in the plane. We also have bijective projections from both these sets to $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ defined as the set of all ordered pairs (x, y) where x and y are real numbers. These functions project points to their coordinates.
The vector is a particularly important concept in mathematics and vector arithmetic along with geometric interpretations of vectors should be studied carefully.
Polar coordinates
Let $\mathbf{v} = (x, y) \neq (0, 0)$ be vectors with length r, so that

$$r = |\mathbf{v}| = \sqrt{x^2 + y^2}.$$

The *angular coordinate* **v** is the angle from the x-axis to the



line through O parallel to $\mathbf{v}$. If we denote this angle by θ, then

$$x = r\cos\theta,$$
$$y = r\sin\theta.$$

Vector arithmetic

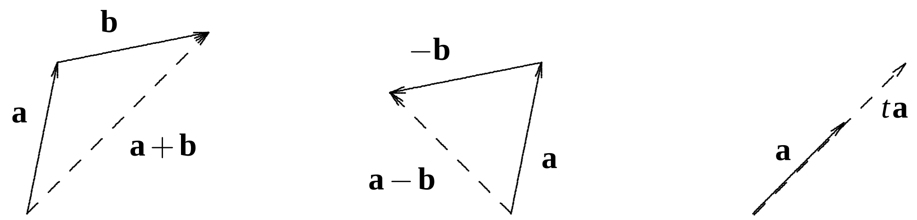
Two operations are available on the set of all vectors in the plane: *vector addition* and *multiplication of a vector by a number*. Let $\mathbf{a} = (a_1, a_2)$ and $\mathbf{b} = (b_1, b_2)$ be vectors and $t \in \mathbb{R}$ be a real number. The *sum* $\mathbf{a}+\mathbf{b}$ is defined by

$$\mathbf{a}+\mathbf{b} = (a_1+b_1, a_2+b_2)$$

and the *product* $t\mathbf{a}$ is defined by

$$t\mathbf{a} = (ta_1, ta_2).$$

These quantities have geometric interpretaions.



Some rules of arithmetic apply and these can be derived by using the corresponding rules for real numbers:

$(\mathbf{a}+\mathbf{b})+\mathbf{c} = \mathbf{a}+(\mathbf{b}+\mathbf{c})$	*(associative law of addition),*
$(st)\mathbf{a} = s(t\mathbf{a})$	*(associative law of multiplication),*
$\mathbf{a}+\mathbf{b} = \mathbf{b}+\mathbf{a}$	*(commutative law of addition),*
$t(\mathbf{a}+\mathbf{b}) = t\mathbf{a}+t\mathbf{b}$	*(distributive law)*
$(s+t)\mathbf{a} = s\mathbf{a}+t\mathbf{a}$	*(distributive law),*
$\mathbf{a}+\mathbf{0} = \mathbf{a}$	*($\mathbf{0}$ is the additive unit),*
$1\mathbf{a} = \mathbf{a}$	*(1 is the multiplicative unit).*

Example 5.1.1

a) Find the polar coordinates of $(4,3)$.

Solution: We can see that the angle in radians is on the interval $]0,\pi/2[$. Then we can find the angle by $\theta = \arctan(3/4) \approx 0,6435\,\text{rad}$. The length is then $r = \sqrt{4^2+3^2} = 5$.

b) Find ordinary coordinates for vector with polar coordinates $|r| = 5,\ \theta = \frac{\pi}{3}\text{rad}$.

Solution: The coordinates are of the form $(r\cos(\theta), r\sin(\theta))$ giving us the coordinates $(5\cos(\pi/3), 5\sin(\pi/3)) = (5\cdot 1/2, 5\cdot\sqrt{3}/2) = (\frac{5}{2}, \frac{5\sqrt{3}}{2})$.

Example 5.1.2

Calculate $(1,2)+5(-2,5)$.

Solution: We start by multiplying and then add upp coordinate for coordinate:

$$(1,2)+5(-2,5) = (1,2)+(-10,25) = (-9,27).$$

Inner products and angles between vectors

The *angle between the vectors* $\mathbf{a}$ and $\mathbf{b}$ is defined by first translating them to an common point of origin and drawing an infinite halfline from each. The angle between the halflines defines the angle between $\mathbf{a}$ and $\mathbf{b}$

The *inner product* is an operation on vectors which assigns to two vectors $\mathbf{a} = (a_1, a_2)$ and $\mathbf{b} = (b_1, b_2)$ a real number

$$\mathbf{a}\cdot\mathbf{b} = a_1 b_1 + a_2 b_2.$$

The main rules of arithmetic for inner products are:

$$\begin{aligned} &\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a} && \textit{(commutative law),}\\ &\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) = \mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{c} && \textit{(distributive law),}\\ &t(\mathbf{a}\cdot\mathbf{b}) = (t\mathbf{a})\cdot\mathbf{b} = \mathbf{a}\cdot(t\mathbf{b}) && \textit{(associative law).} \end{aligned}$$

Some familiar rules also apply for combining inner products and lengths of vectors:

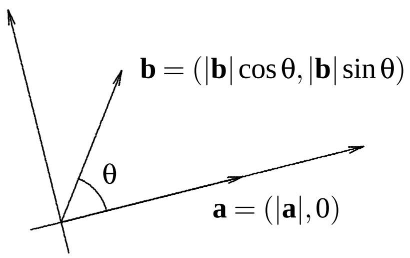
$$\begin{aligned} |\mathbf{a}|^2 &= \mathbf{a}\cdot\mathbf{a},\\ |\mathbf{a}+\mathbf{b}|^2 &= |\mathbf{a}|^2 + |\mathbf{b}|^2 + 2\mathbf{a}\cdot\mathbf{b},\\ |\mathbf{a}-\mathbf{b}|^2 &= |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2\mathbf{a}\cdot\mathbf{b}. \end{aligned}$$

The above rules can easily be derived by using the definition of the inner product. Consider the last of the above rules. By isolating $\mathbf{a}\cdot\mathbf{b}$ we obtain

$$\mathbf{a}\cdot\mathbf{b} = \frac{1}{2}(|\mathbf{a}|^2 + |\mathbf{b}|^2 - |\mathbf{a}-\mathbf{b}|^2).$$

The right hand side of the equation contains only length of vectors and it is clear that these do not change if we translate the coordinate system or rotate it around a point. The following must therefore hold:

The inner product of two vectors is unchanged even if the coordinate system is translated or rotated.



Now consider what this result implies. Take two vectors, $\mathbf{a}$ and $\mathbf{b}$ and let θ be the angle between the two. Since the inner product is independent of translating or rotating the coordinate system, we can choose the coordinate system so that $\mathbf{a}$ is on the x-axis. In this coordinate system $\mathbf{a} = (|\mathbf{a}|, 0)$, θ is the angular coordinate of $\mathbf{b}$ and hence $\mathbf{b} = (|\mathbf{b}|\cos\theta, |\mathbf{b}|\sin\theta)$. This yields a useful rule for inner products:

$$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta.$$

This result provides yet another property of the inner product:

The vectors $\mathbf{a}$ *and* $\mathbf{b}$ *are orthogonal if and only if* $\mathbf{a}\cdot\mathbf{b} = 0$.

Example 5.2.1
a) What is the dot product of $(1,1)$ and $(2,1)$?
Solution: We calculate $(1,1)\cdot(2,1) = 1\cdot 2 + 1\cdot 1 = 2 + 1 = 3$.
b) Calculate $(1,2)\cdot((-2,1)-(12,0))$.
Solution: We just calculate:

$$(1,2)\cdot((-2,1)-(12,0)) = (1,2)\cdot(-2,1) - (1,2)\cdot(12,0) = 0 - 12 = -12.$$

Example 5.2.2
Find the cosine of the angle θ between $(1,2)$ and $(2,1)$.

Solution: We use the relationship $(1,2)\cdot(2,1) = |(1,2)||(2,1)|\cos(\theta)$ and by isolating we get

$$\cos(\theta) = \frac{(1,2)\cdot(2,1)}{|(1,2)||(2,1)|} = \frac{2+2}{\sqrt{5}\cdot\sqrt{5}} = \frac{4}{5}.$$

Triangular inequality

In a triangle every side length is less that or equal to the sum of the other two. One of the best well known inequalities of mathematics is the triangle inequality, which describes this fact in terms of vectors. Consider how this can be derived in terms of known equations. First note that according to the previous section

$$|\mathbf{a}+\mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 + 2\mathbf{a}\cdot\mathbf{b}.$$

Now rewrite the sum of two squares as follows

$$(|\mathbf{a}|+|\mathbf{b}|)^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 + 2|\mathbf{a}||\mathbf{b}|.$$

We know that $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$ and since $\cos\theta \leq 1$ we obtain $\mathbf{a}\cdot\mathbf{b} \leq |\mathbf{a}||\mathbf{b}|$. Using the above equations we see that

$$|\mathbf{a}+\mathbf{b}|^2 \leq (|\mathbf{a}|+|\mathbf{b}|)^2.$$

Since both $|\mathbf{a}+\mathbf{b}|$ and $|\mathbf{a}|+|\mathbf{b}|$ are positive quantities we have shown the following:

Triangular inequality. For any two vectors **a** and **b** satisfy the inequaltiy

$$|\mathbf{a}+\mathbf{b}| \leq |\mathbf{a}|+|\mathbf{b}|.$$

with equality if and only if **a** and **b** have the same direction, $\mathbf{a} = t\mathbf{b}$ or $\mathbf{b} = t\mathbf{a}$ where $t \leq 0$.

Other inequalities can be derived from the triangular inequality. Note that

$$\begin{aligned}|\mathbf{a}| &= |\mathbf{a}-\mathbf{b}+\mathbf{b}| = |(\mathbf{a}-\mathbf{b})+\mathbf{b}|\\ &\leq |\mathbf{a}-\mathbf{b}|+|\mathbf{b}|,\end{aligned}$$

and hence

$$|\mathbf{a}|-|\mathbf{b}| \leq |\mathbf{a}-\mathbf{b}|.$$

For any vector **a** we know that $|\mathbf{a}| = |-\mathbf{a}|$ and in particular $|\mathbf{a}-\mathbf{b}| = |-(\mathbf{a}-\mathbf{b})| = |\mathbf{b}-\mathbf{a}|$. We see that

$$|\mathbf{b}|-|\mathbf{a}| \leq |\mathbf{b}-\mathbf{a}| = |\mathbf{a}-\mathbf{b}|.$$

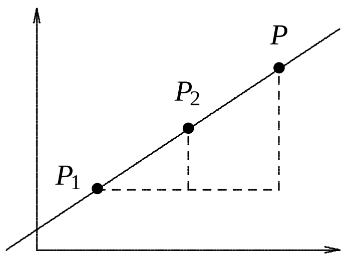
By combining the last two inequalities we obtain a new result.

For any two vectors **a** and **b** we have

$$||\mathbf{a}|-|\mathbf{b}|| \leq |\mathbf{a}-\mathbf{b}|.$$

The equation of a line and a circle

A *line* ℓ in a plane is uniquely determined by two points. Consider two distinct points in a plane with a given coordinate system and write them as $P_1 = (x_1, y_2)$ and $P_2 = (x_2, y_2)$. If the line is *vertical*, then $x_1 = x_2$ and if it is *horizontal*, then $y_1 = y_2$. If the line is not vertical, then the number



$$h = \frac{y_2 - y_1}{x_2 - x_1}$$

is *well defined* (i.e. the number exists and is uniquely defined) and is called the *slope* of the line. By using the property of ratios between lengths in similar triangles we see that the slope does not depend on which two points on the line are chosen. Now consider an arbitrary point $P = (x,y)$ on the line to obtain

$$h = \frac{y - y_1}{x - x_1} \qquad \Leftrightarrow \qquad y = y_1 + h(x - x_1)$$

The latter equation and all equivalent equations are the *equation of the line* ℓ. Note that the horizontal line gets the equation $y = y_1$ and the vertical line is described by the equation $x = x_1$.

The equation of a line can always be written in the standardized form

$$ax + by + c = 0 \qquad \text{with } (a,b) \neq (0,0).$$

The *normal vector* $\mathbf{n} = (b, -a)$ has a specific meaning as it is orthogonal to the line. If it also has unit length, $|\mathbf{n}| = 1$, it is called a *unit normal vector*.

If a line ℓ is described by the equation $ax + by + c = 0$ and $P_0 = (x_0, y_0)$ is a given point in the plane, then its distance from the line is defined as the smallest distance between the given point and any point on the line. This is given by

$$\frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}.$$

A *circle* centered at the point M and *radius* $r > 0$ consists of all points P such that the distance $|MP|$ between M and P is equal to r, $|MP| = r$. If $M = (p,q)$ and $P = (x,y)$, then this equation is $|MP| = \sqrt{(x-p)^2 + (y-q)^2} = r$ and is equivalent to

$$(x-p)^2 + (y-q)^2 = r^2.$$

Example 5.4.1

A line goes through the points $(2,0)$ and $(5,-2)$.

a) Find the line's slope.

Solution: The slope is given by the ratio:

$$\frac{y_1 - y_2}{x_1 - x_2} = \frac{0 - (-2)}{2 - 5} = -\frac{2}{3}$$

b) Find the equation of the line and write it on standardized form.

Solution: We found the slope in part **a** so the equation is of the form $y = \frac{-2}{3}x + c$, where c is a constant we need to find. By putting $(2,0)$ into the equation we get $0 = 2\frac{-2}{3} + c$ which gives $c = \frac{4}{3}$. We then rewrite to get the standardized form:

$$\frac{2}{3}x + y - \frac{4}{3} = 0.$$

c) Find the distance of $(-1,3)$ from the line.
Solution: Now, having already found the standardized equation we can use a given formula for distance D directly.

$$D = \frac{|\frac{2}{3}\cdot(-1)+1\cdot 3-\frac{4}{3}|}{1^2+\frac{2^2}{3^2}} = \frac{1}{13/9} = \frac{9}{13}.$$

Example 5.4.2
Find the equation of a ring with radius 3 and center $(2,-1)$.
Solution: A circle with middle in (x_0,y_0) and radius r has the equation $(x-x_0)^2+(y-y_0)^2 = r^2$. Thus the equation is $(x-2)^2+(y+1)^2 = 3^2 = 9$.

24 Domain and image, injections and surjections

25 Composite functions

26 Limits

26.1 Limits

26.1.1 Handout

Limits

Let $f : D_f \to \mathbb{R}$ be a real-valued function which is defined on a subset D_f of $\mathbb{R}$ and let a be an *accumulation point* of the set D_f. The word *accumulation point* means that for any open interval J containing a there is a point $x \neq a$ which is in $D_f \cap J$. Note that if D_f is an open interval $D_f =]\alpha,\beta[$, where $\alpha < \beta$, then all points in the closed interval $[\alpha,\beta]$ are accumulation points of D_f.
There are several types of limits

1)	$\lim_{x\to a} f(x)$	*the limit of $f(x)$ as x goes to a*
2)	$\lim_{x\to a+} f(x)$	*the limit of $f(x)$ as x goes to a from the right*
3)	$\lim_{x\to a-} f(x)$	*the limit of $f(x)$ as x goes to a from the left*
4)	$\lim_{x\to +\infty} f(x)$	*the limit of $f(x)$ as x goes to plus infinity*
5)	$\lim_{x\to -\infty} f(x)$	*the limit of $f(x)$ as x goes to minus infinity*

In 4) it is assumed that D_f contains points in every half-open interval $]c,+\infty[$ and in 5) it is assumed that D_f contains points in every half-open interval $]-\infty,c[$.
These limits describe the behaviour of the function f either close to the point a or as x goes to $\pm\infty$.
Formally, *$f(x)$ converges to the number L as x goes to a*, if for every open interval I containing L there is an open interval J wich contains a such that $f(x) \in I$ for all $x \in (J \cap D_f)\setminus\{a\}$. Since every open interval which contains L will contain all symmetrics intevals $]L-\varepsilon, L+\varepsilon[$ for small enough $\varepsilon > 0$ and every open interval which contains a contains all symmetric intervals $]a-\delta,a+\delta[$ with $\delta > 0$ small enough, we see that $f(x)$ converges to L as x goes to a if and only if for every $\varepsilon > 0$ there is a $\delta > 0$ such that $|f(x)-L| < \varepsilon$ for all $x \in D_f$ with $0 < |x-a| < \delta$.
When this holds the number L is called the *limit of the function f as x goes to a* and is denoted by $\lim_{x\to a} f(x)$. This is also denoted by $f(x) \to L$ if $x \to a$.
We will also permit $+\infty$ and $-\infty$ to be permissible limit. The definition for the $+\infty$ cases are that *$f(x)$ goes to $+\infty$ (plus infinity) as x goes to a*, if for every real number A there is an

open interval J which contains a such that $f(x) > A$ for every $x \neq a$ in $D_f \cap J$. If this holds we write

$$\lim_{x \to a} f(x) = +\infty$$

and say that the *limit of $f(x)$ as x goes to a is plus infinity.*
A corresponding definition is used when the limit of $f(x)$ is $-\infty$.
Let f and g be two functions on I, let $a \in I$ and suppose the limits

$$\lim_{x \to a} f(x) \qquad \text{og} \qquad \lim_{x \to a} g(x)$$

exist neither is $\pm\infty$. Then

$$\lim_{x \to a} (f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$$

and

$$\lim_{x \to a} (f(x)g(x)) = \big(\lim_{x \to a} f(x)\big)\big(\lim_{x \to a} g(x)\big).$$

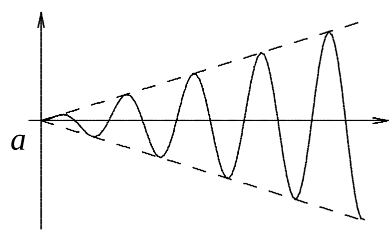
Further, if $\lim_{x \to a} g(x) \neq 0$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$$

Now suppose we have functions $f : D_f \to \mathbb{R}$, and $g : D_g \to \mathbb{R}$ defined on the sets D_f and D_g and that $f(D_f) \subset D_g$, so that the composite function $g \circ f$ is well defined. If $\lim_{x \to a} f(x) = b$ and $\lim_{y \to b} g(y) = c$ exist, then

$$\lim_{x \to a} (g \circ f)(x) = \lim_{x \to a} g(f(x)) = c.$$

The squeeze theorem
The *squeeze theorem* is a useful method for computing the limit of a function. The theorem states that if f, g and h are three functions with values in $\mathbb{R}$ such that $f \leq g \leq h$ and $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$, then $\lim_{x \to a} g(x) = L$.



Example: The limit $\lim_{x \to 0} \sin x / x = 1$
By drawing the unit circle and the point $(\cos x, \sin x)$, where $0 < x < \frac{1}{2}\pi$, we see the two inequalities

$$\sin x < x < \tan x = \frac{\sin x}{\cos x}$$

from which we note that

$$\cos x \leq \frac{\sin x}{x} \leq \frac{1}{\cos x}.$$

by using the squeeze theorem and the fact that $\cos x \to 1$ ef $x \to 0$, we have

$$\lim_{x \to 0} \frac{\sin x}{x} = 1.$$

Example 8.1.1
Find the limit $\lim_{x \to 1} \frac{x^2+1}{x^2+2x-1}$.
Solution: The denominator is not zero for $x = 1$ so the function is continuous there. Then we can without trouble conclude $\lim_{x \to 1} \frac{x^2+1}{x^2+2x-1} = \frac{1+1}{1+2-1} = 1$.

Example 8.1.2
Find the limit $\lim_{x\to 2} \frac{x^2-4}{x+2}$.
Solution: The limits over and under are both 0. Notice that

$$\lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2} \frac{(x+2)(x-2)}{(x-2)} = \lim_{x\to 2} x+2 = 4.$$

Example 8.1.3
Find the limit $\lim_{x\to\infty} \frac{\sin(x)}{x}$ if it exists.
Solution: We know that $|\sin(x)| \leq 1, \forall x \in \mathbb{R}$ so

$$\lim_{x\to\infty} \left|\frac{\sin(x)}{x}\right| \leq \lim_{x\to\infty} \left|\frac{1}{x}\right| = 0$$

so by the squeeze theorem we find $\lim_{x\to\infty} \frac{\sin(x)}{x} = 0$.

Example 8.1.4
Find the limit $\lim_{x\to\infty} \frac{x^2+1}{2x^2+5x+1}$ if it exists.
Solution: We can divide through by x^2 and get

$$\lim_{x\to\infty} \frac{x^2+1}{2x^2+5x+1} = \lim_{x\to\infty} \frac{1+1/x^2}{2+5/x+1/x^2} = \frac{\lim_{x\to\infty} 1+1/x^2}{2+5/x+1/x^2} = \frac{1}{2}.$$

Example 8.1.5
find the limit $\lim_{x\to 1^+} \frac{x^2+2x-1}{x^2-1}$ if it does exist.
Solution: For $x=1$ the polynomial above takes the value 2 but the one below the bar has value zero, but it takes positive values on $]1,+\infty[$ so we get the limit $\lim_{x\to 1^+} \frac{x^2+2x-1}{x^2-1} = +\infty$. Notice that the limit from the left is $\lim_{x\to 1^-} \frac{x^2+2x-1}{x^2-1} = -\infty$.

27 Asymptotes and limits; rational functions

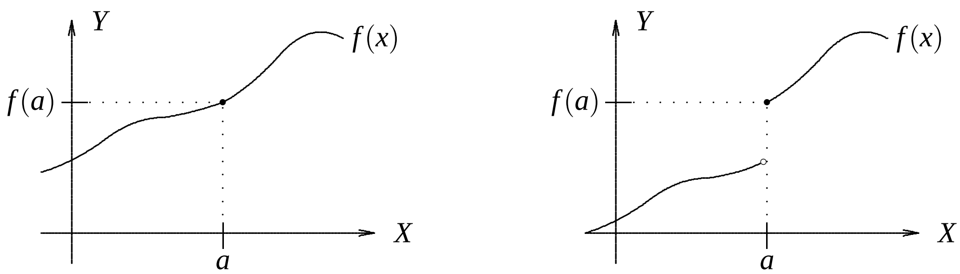
28 Continuity

28.1 Continuity

28.1.1 Handout

Definition of continuity
Let $f : D_f \to \mathbb{R}$ be defined on a subset of $\mathbb{R}$, $a \in D_f$ and assume there is an open interval within D_f which contains a. The function f is *continuous at the point a* if

$$\lim_{x\to a} f(x) = f(a).$$



f is continuous at a $\qquad$ f discontinuous at a

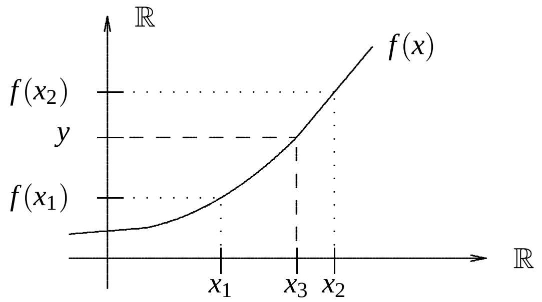
The function f is said to be continous if it is continuous at all points in D_f.
If the functions f and g are defined on the same set and continuous at the point a, then the functions $f+g$ and fg are continuous at a and we obtain

$$\lim_{x\to a}(f(x)+g(x)) = f(a)+g(a) \qquad \text{og} \qquad \lim_{x\to a}(f(x)g(x)) = f(a)g(a).$$

If in addition $g(a) \neq 0$, then f/g is continuous at a and

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{f(a)}{g(a)}.$$

Composite functions
If the function values of g are in the domain of f so the composite $f \circ g$ is defined, and both are continuous, then the composite function is continuous.
Intermediate value theorem
If $f : D_f \to \mathbb{R}$ is a continuous function on a subset of $\mathbb{R}$ sem with values $f(x_1)$ at the point x_1 and $f(x_2)$ at x_2 og $f(x_1) \neq f(x_2)$, then for all y between $f(x_1)$ and $f(x_2)$ there is $x_3 \in]x_1,x_2[$ such that $f(x_3) = y$.



Extrema
Let $f : D_f \to \mathbb{R}$ be a continuous function on a closed and finite interval $D_f \subset \mathbb{R}$. Then f takes on a largest and a smallest value in the interval, i.e. there are $x_1, x_2 \in D_f$ such that for all x in D_f one has $f(x_1) \leq f(x) \leq f(x_2)$.
Polynomials and rational functions
If f is the constant function c, $f(x) = c$ for all $x \in \mathbb{R}$, then $\lim_{x\to a} f(x) = c = f(a)$, and hence all constant functions are continuous. The function $f(x) = x$ satisfies $\lim_{x\to a} f(x) = a = f(a)$ and is therefore also continuous. Using induction we can also see that all power functions $x \mapsto x^m$ are continuous for $m \in \mathbb{N}_0$. Similarly, every polynomial

$$P(x) = a_m x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + a_0$$

is continuous on $\mathbb{R}$.
If P and Q are polynomials, a function $f = P/Q$ is a *rational function*. From results on the ratio of continuous functions we see that every rational function is continuous at points which are not zeroes of the denominator Q, but those points are termed the *poles* of the function f. In particular, all power functions of the form $x \mapsto x^{-m} = 1/x^m$, $m \in \mathbb{N}$, are continuous on $\mathbb{R} \setminus \{0\}$.
Monotonic functions
A function f defined on an interval I is said to be:

(i) *increasing*, if $x_1 < x_2$ implies $f(x_1) \leq f(x_2)$,

(ii) *strictly increasing*, if $x_1 < x_2$ implies $f(x_1) < f(x_2)$,

(iii) *decreasing* if $x_1 < x_2$ implies $f(x_1) \geq f(x_2)$ and

(iv) *strictly decreasing* if $x_1 < x_2$ implies $f(x_1) > f(x_2)$,

(v) *monotonic* if it is increasing or decreasing

(vi) *strictly monotonic* if it is strictly increasing or strictly decreasing.

Every strictly monotonic function is one-to-one.
If the function f is monotonic and continuous then the the er einhalla og samfellt, the intermediate value theorem implies that the funciton values form an interval:
Let $f : [a,b] \to \mathbb{R}$ be a strictly increasing [decreasing], continuous functions and set $c = f(a)$, $d = f(b)$ [$c = f(b)$, $d = f(a)$], then f is a bijective function of $[a,b]$ onto the interval $[c,d]$ with inverse $f^{-1} : [c,d] \to [a,b]$ which is strictly increasing [decreasing].

Power functions with a rational power

Let $r = p/q$ be a rational number so $q \neq 0$ and consider the function $f(x) = x^r$ defined on the positive reals $\mathbb{R}_+^* = \{x \in \mathbb{R}; x > 0\}$. Then f is strictly increasing if $r > 0$ and strictly decreasing if $r < 0$. The root $x \mapsto x^{\frac{1}{q}}$ is the inverse of the continuous power function $x \mapsto x^q$ and is therefore a continuous function. The function $x \mapsto x^p$ is also continuous for all values of p. The function f is the composite of these: $f(x) = (x^{\frac{1}{q}})^p$.

Trigonometric functions

Consider now the trigonometric functions, with cos as a specific example. Take a number $a \in \mathbb{R}$ and write $h = x - a$. First recall the angle sum formula for cos which gives

$$\cos h = \cos(\tfrac{1}{2}h + \tfrac{1}{2}h) = (\cos\tfrac{1}{2}h)^2 - (\sin\tfrac{1}{2}h)^2 = 1 - 2\sin^2(\tfrac{1}{2}h).$$

Now apply the same formula to $\cos(a+h)$ to obtain

$$\begin{aligned}\cos x - \cos a &= \cos(a+h) - \cos a = \cos a \cos h - \sin a \sin h - \cos a\\ &= \cos a(\cos h - 1) - \sin a \sin h\\ &= -2\cos a \sin^2(\tfrac{1}{2}h) - \sin a \sin h\end{aligned}$$

Since $|\cos a| \leq 1$ and $|\sin a| \leq 1$, we have the inequality

$$|\cos x - \cos a| \leq 2\sin^2(\tfrac{1}{2}h) + |\sin h|.$$

By drawing the unit circle and looking at the point $(\cos t, \sin t)$ we see that the inequality $|\sin t| \leq |t|$ holds for $t \in \mathbb{R}$. This implies that

$$|\cos x - \cos a| \leq 2|h| = 2|x - a|.$$

We will proceed to use this inequality to prove that cos is a continuous function at the point a by using the definition of a limit. Hence take $\varepsilon > 0$ and choose $\delta = \frac{1}{2}\varepsilon$. For every x which satisfies $|x - a| < \delta$ we now know that

$$|\cos x - \cos a| \leq 2|x - a| = 2\delta = \varepsilon.$$

This proves that cos is continuous at a and since a was arbitrary, this holds for all real numbers so cos is a continuous functions on $\mathbb{R}$.
Next note that $\sin x = \cos(\frac{1}{2}\pi - x)$ so sin is a composition of two continuous functions, cos and $x \mapsto \frac{1}{2}\pi - x$. Hence sin is also a continuous function. Finally, $\tan x = \sin x / \cos x$ and $\cot x = \cos x / \sin x$ are ratios of continuous functions so tan is continuous at all points a where $\cos a \neq 0$ and cot is continuous at all points a where $\sin a \neq 0$.
The conclusion is that all the trigonometric functions, cos, sin, tan and cot are continuous in their domains.

Exponential and logarithmic functions
Let $a > 0$ be a positive real number. The *exponential function with base a* is the function defined on $\mathbb{R}$ with $x \mapsto a^x$. This function is strictly increasing if $a > 1$ and strictly decreasing if $a < 1$. In both cases is the image the set of positive real numbers, $\{x \in \mathbb{R}; x > 0\}$. The inverse therefore exists. It is denoted $\log_a$ and called the *logarithm with base a*. These functions are continuous.
For the special case, $a = 1$ the functions $x \mapsto a^x$ is the constant function 1. In all cases we see that the continuity imples that the power formulas

$$a^{x+y} = a^x a^y \qquad \text{og} \qquad (a^x)^y = a^{xy}$$

generalize so that they apply for all $x, y \in \mathbb{R}$. If we take two positive real numbers s and t and write these as $s = a^x$ and $t = a^y$, then $x = \log_a s$ and $y = \log_a t$ and the first power formula gives

$$\log_a(st) = \log_a(a^x a^y) = \log_a(a^{x+y}) = x + y = log_a s + \log_a t.$$

If we take a real number r, then $s^r = (a^x)^r = a^{rx}$. The second power formula implies

$$\log_a(s^r) = \log_a(a^{rx}) = rx = r\log_a s.$$

We have thus derived the following properties of logarithms:

$$\log_a(st) = \log_a s + \log_a t \qquad \text{og} \qquad \log_a s^r = r\log_a s,$$

which hold for all positive real numbers s and t and all real numbers r.
Example 8.2.1
Find where the function $f(x) := \frac{x^2-2x-1}{x^2-1}$ is continuous.
Solution: We know the denominator has zeros in 1 and -1 and the numerator does not have zeros there. Then the function is convergent in all real numbers except 1 and -1.

Example 8.2.2
Let $f : \mathbb{R} \to \mathbb{R}$; $f(x) = \begin{cases} |x|, \ x > 0 \\ \sin(x), \ x < 0 \end{cases}$. Is it possible to give f value in $x = 0$ such that the function becomes continuous there?
Solution: we start by finding the limits from the left and right in zero and check if they are the same. We have $\lim_{x\to 0^-} \sin(x) = 0$ and $\lim_{x\to 0^+} |x| = 0$ so by defining $f(0) := 0$ f becomes continuous in zero.

Example 8.2.3
Find the intervals where $f : \mathbb{R} \to \mathbb{R}$; $f(x) = ||x+3| - 5|$ is monotonic, and tell if it is increasing or decreasing.
Solution: By splitting into cases we can see that

$$f(x) = \begin{cases} |x+3| - 5, \ |x+3| \geq 5 \\ 5 - |x+3|, \ |x+3| < 5 \end{cases}.$$

We can see that the slope changes where $|x+3| = 5$ which is in -8 and 2, also where $x+3 = 0$ which is in $x = -3$. Then f is decreasing on $]-\infty, -8[$, increasing on $]-8, -3[$, decreasing on $]-3, 3[$ and increasing on $]3, +\infty[$.

Example 8.2.4
Simplify

$$\log_b\left(\left(\log_a(a^{xb})\right)^y\right).$$

Solution: We calculate the innermost brackets by the definition of logarithm:

$$\log_b\left(\left(\log_a(a^{xb})\right)^y\right) = \log_b\left((xb))^y\right)$$

Then we use known properties of logarithm

$$= y\log_b(xb) = y(1+\log_b(x)).$$

Example 8.2.5
Where is the function $f : \mathbb{R} \to \mathbb{R},\ f(x) := \log_5\left(10^{\sin(x^2-5)}\right)$ continuous?
Solution: The polynomial $x^2 - 5$ is continuous on all $\mathbb{R}$ and so is sin and 10^x which takes only positive values. $\log_5(x)$ is continuous for all positive x. Then f is composed of continuous functions and therefore continuous on all $\mathbb{R}$.

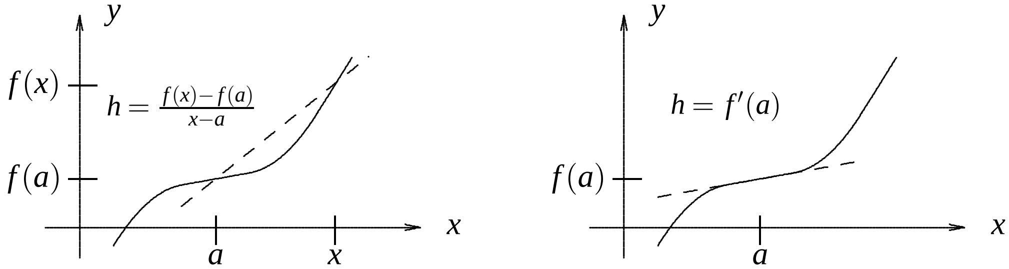
29 Derivatives

29.1 Derivatives

29.1.1 Handout

Consider a function $f : I \to \mathbb{R}$ defined on an open interval I on the real axis and consider two points on its graph, $(a, f(a))$ and $(x, f(x))$. The line through these two points is the *secant line to the graph of f through $(a, f(a))$ and $(x, f(x))$*. The slope of the secant line is

$$\frac{f(x)-f(a)}{x-a}.$$



The function f is said to be *differentiable* at the point a if the limit

$$f'(a) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a}$$

exists. The value $f'(a)$ is the *derivative of the function f at the point a*. If the function f is differentiable in every point in the interval I, we call f a *differentiable* function and the derivative f' is a function on I. The process of finding a derivative is *differentiation*.
Note that by writing $h = x - a$ we can also define the derivative by the formula

$$f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}.$$

A line with the equation

$$y = f(a) + f'(a)(x-a)$$

is the *tangent to the graph of f* at the point a.
If the function f is differentiable in a point a then it is continuous in a.

29.1.2 Higher order derivatives

If f is a continuous function then in some cases the derivative f' itself may be differentiable. In this case one may differentiate again to again to obtain the *second derivative* $f'' = (f')'$. This function may also be differentiable and thus we may continue as long as the resulting function is differentiable.
Once we have derivatives of f of more then the second or third order, i.e. more than . f'' or f''', new notation is needed. Therefore, when the function f is differentiated n times where $n \geq 4$ the result is denoted $f^{(n)}$.
Example 9.1.1
Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) := x^2 - 2x + 1$. Find the slope of a line through the points $(x, f(x))$ and $(x+h, f(x+h))$ and use that to find the derivative.
Solution: The lines' slope is

$$\frac{f(x+h) - f(x)}{x+h-x} = \frac{(x+h)^2 - 2(x+h) + 1 - x^2 + 2x - 1}{h} = \frac{x^2 + 2hx + h^2 - 2h - x^2}{h} = 2x + h - 2.$$

Then the derivative is $\lim_{h \to 0} 2x + h - 2 = 2x - 2$.

Example 9.1.2
Find the third derivative of $f(x) := x^3 + 2x^2 - 5x + 258$.
Solution: We differentiate repeatetly:

$$f'(x) = 3x^2 + 4x - 5, \quad f'' = 6x + 4, \quad f''' = 6.$$

Now let f and g be functions on an interval I and suppose they are both differentiable at the point a. Then $f+g$ is also differentiable at a and

$$(f+g)'(a) = f'(a) + g'(a).$$

Similarly fg is differentiable at a with

$$(fg)'(a) = f'(a)g(a) + f(a)g'(a)$$

This last formula is called *Leibniz's law*. If $g(a) \neq 0$, then f/g is also differentiable at a with

$$\left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{g(a)^2}.$$

Next suppose $f : I \to \mathbb{R}$ and $g : J \to \mathbb{R}$ be such that $f(I) \subseteq J$. If the function f is differentiable at a and g is differentiable at $b = f(a)$, then the compositions $g \circ f$ is differentiable at a with

$$(g \circ f)'(a) = g'(f(a)) \cdot f'(a).$$

This rule is commonly called the *chain rule*.
If f is a strictly monotonic and continuous funciton on the interval I then f has an inverse f^{-1}. Suppose $a \in I$ and write $b = f(a)$. If f is differentiable at a, then f^{-1} is differentiable at the point b with

$$(f^{-1})'(b) = \frac{1}{f'(a)}.$$

Example 9.2.1
Let $f : \mathbb{R} \to \mathbb{R}$; $f(x) := x^2 - 2x + 1$ and $g : \mathbb{R} \to \mathbb{R}$; $g(x) = x + 5$. find the derivative of $\frac{f}{g}$.

Solution: We first find the derivatives for f and g. $f'(x) = 2x-2$ and $g'(x) = 1$. We then use a rule of arithmetic for derivative of rational functions:

$$\left(\frac{f}{g}(x)\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)} = \frac{(2x-2)(x+5) - (x^2-2x+1)1}{(x+5)^2}$$
$$= \frac{2x^2+10x-2x-10-x^2+2x-1}{(x+5)^2} = \frac{x^2+10x-11}{(x+5)^2}.$$

Consider some specific examples of functions and their derivatives.

(1) The constant function $f(x) = c$ is differentiable at all points and $f'(a) = 0$ for all $a \in \mathbb{R}$. This is easily seen from the definition of the derivative, namely:

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \frac{c-c}{h} = 0.$$

(2) The function $f(x) = x$ is differentiable at every point and $f'(a) = 1$ for all $a \in \mathbb{R}$. This is also seen from the definition of the derivative since $f(a) = a$ and $f(a+h) = a+h$:

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \frac{a+h-a}{h} = \lim_{h\to 0} 1 = 1.$$

(3) Any power function $f(x) = x^n$ with the power n a natural number is differentiable with derivative $f'(a) = na^{n-1}$. This can be shown using a combination of induction and Leibniz's law, or using the binomial theorem:

$$(a+h)^n = a^n + na^{n-1}h + \binom{n}{2}a^{n-2}h^2 + \cdots + \binom{n}{n-1}ah^{n-1} + h^n$$
$$= a^n + h\big(na^{n-1} + \binom{n}{2}a^{n-2}h + \cdots + h^{n-1}\big),$$

where $\binom{n}{k}$ denotes the binomial coefficients. It follows that

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \frac{(a+h)^n - a^n}{h}$$
$$= \lim_{h\to 0} \big(na^{n-1} + \binom{n}{2}a^{n-2}h + \cdots + h^{n-1}\big) = na^{n-1}.$$

(4) All polynomials $f(x) = a_m x^m + \cdots + a_1 x + a_0$ are differentiable at all points and from the rules for sums and products we obtain $f'(x) = ma_m x^{m-1} + \cdots + 2a_2 x + a_1$

(5) Every rational function $f(x) = P(x)/Q(x)$, where P and Q are polynomials, are differentiable at all points $a \in \mathbb{R}$, where $Q(a) \neq 0$ and the derivative can be computed using the rule for computing the derivative of a ratio.

(6) The trigonometric functions cos and sin are differentiable everywhere with derivatives

$$\cos' x = -\sin x \qquad \text{og} \qquad \sin' x = \cos x.$$

The function tan and cot are also differentiable and the rule for the derivative of a ration gives

$$\tan' x = \frac{1}{\cos^2 x} = 1 + \tan^2 x \qquad \text{og} \qquad \cot' x = \frac{-1}{\sin^2 x} = -(1+\cot^2 x).$$

(7) The exponential function $f(x) = \exp x = e^x$ is differentiable everywhere with $f'(x) = f(x) = e^x$ for all $x \in \mathbb{R}$,

$$\exp'(x) = \exp(x).$$

Since the natural logarithm is the inverse of the exponential function, its derivative is $\ln'(x) = 1/x$ for $x > 0$.

(8) The exponential function with base $a > 0$, $f(x) = a^x$, is differentiable on the real line $\mathbb{R}$ with derivative

$$f'(x) = (\ln a)a^x = (\ln a) f(x).$$

Example 9.3.1
Let $f : \mathbb{R}_+ \to \mathbb{R}$, $f(x) := \ln(x)e^x$. Calculate $f'(x)$.
Solution: We use Leibniz' law, we have the derivatives $\ln'(x) = \frac{1}{x}$ and $\exp'(x) = \exp(x)$ and by that we get

$$f'(x) = \ln'(x)\exp(x) + \ln(x)\exp'(x) = \frac{e^x}{x} + \ln(x)e^x.$$

Example 9.3.2
Let $f : \mathbb{R} \to \mathbb{R}$; $f(x) := \sin(2x)$, $g : \mathbb{R} \to \mathbb{R}$; $g(x) := \frac{1}{x^2}$, $h : \mathbb{R} \to \mathbb{R}$; $h(x) := \cos(x^2)$.
a) Calculate $(gh)'(x)$.
Solution: Here we can use Leibniz' law. We find the derivatives , $g'(x) = \frac{-2}{x^3}$ and by the chain rule we get $h'(x) = -2\sin(x^2)x$. Now putting this into Leibniz' rule we get:

$$(gh)'(x) = g'(x)h(x) + g(x)h'(x) = \frac{-2\cos(x^2)}{x^3} + \frac{-2\sin(x^2)x}{x^2} = \frac{-2\cos(x^2)}{x^3} - \frac{2\sin(x^2)}{x}.$$

b) Calculate $(f+h)'(x)$.
Solution: We get h' from part a) and by the chain rule $f'(x) = 2\cos(2x)$. By computational rules for derivatives we then get

$$(f+h)'(x) = f'(x) + h'(x) = 2\cos(2x) - 2\sin(x^2)x.$$

c) Calculate $\left(\frac{f}{h}\right)'(x)$.
Solution: We have already found the derivatives of f and h. We use that and get

$$\left(\frac{f}{h}\right)'(x) = \frac{f'(x)h(x) - f(x)h'(x)}{h^2(x)} = \frac{2\cos(2x)\cos(x^2) + 2\sin(2x)\sin(x^2)x}{\cos^2(x^2)}.$$

d) Calculate $(f \circ g)'(x)$.
Solution: We know the derivatives of f and g already. By using chain-rule we get

$$(f \circ g)'(x) = f'(g(x))g'(x) = 2\cos\left(\frac{2}{x^2}\right)\frac{-2}{x^3} = \frac{-4\cos(2/x^2)}{x^3}.$$

30 Tangents

31 Derivatives, trigonometric functions, chain rule and implicit differentiation

32 Maxima and minima of functions

32.1 Maxima and minima of functions

32.1.1 Handout

The sign of the derivative of a function gives information on its behavior.

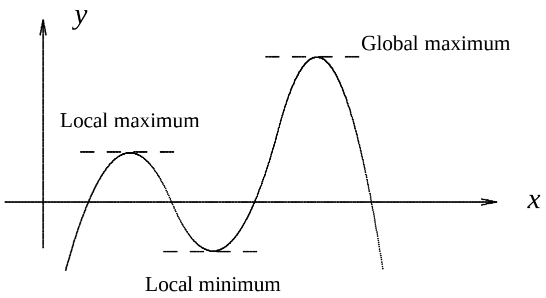
The derivative of a monotone function

Let $f: I \to \mathbb{R}$ be a function define on an open interval I. If f is increasinf on the interval I, $a \in I$ and $h > 0$ then $(f(a+h) - f(a))/h \geq 0$. If f is differentiable at a, then this implies that $f'(a) \geq 0$. For decreasing functions the inequality is reversed and we have:

(i) If f is increasing on I and differentiable at $a \in I$, then $f'(a) \geq 0$.

(ii) If f is decreasing on I and differentiable at $a \in I$, then $f'(a) \leq 0$.

Extrema of a function



We say that the function $f: I \to \mathbb{R}$ has a *local maximum* at the point c if there is an open interval J which contains c such that $f(x) \leq f(c)$ for all $x \in J$ and we say that the function $f: I \to \mathbb{R}$ has a *local minimum* at the point c if there is an open interval J which contains c such that $f(x) \geq f(c)$ for all $x \in J$. The function f has a *local extremum* at the point c if it has either a local maximum or a local minimum at c.

If f has a local maximum at c and $f(x) \leq f(c)$ for all x in I then the maximum is a *global maximum*. Similarly c is a *global minimum* if $f(x) \geq f(c)$ for all x in I and *global extremum* if no distinction is made between a maximum and a minimum.

Now suppose that f has a local mininum at the point c and that we have a open interval which contains a such that $f(x) \geq f(c)$ for all $x \in J$. If $h > 0$ and $c+h \in J$, then $(f(c+h) - f(c))/h \geq 0$ which implies that

$$\lim_{h \to 0+} \frac{f(c+h) - f(c)}{h} \geq 0,$$

if the limit exists. In the other hand if $h < 0$ and $c+h \in J$, then both $f(c+h) - f(c) \leq 0$ and $h < 0$ so that $(f(c+h) - f(c))/h \leq 0$ and we obtain

$$\lim_{h \to 0+} \frac{f(c+h) - f(c)}{h} \leq 0,$$

if the limit exists.

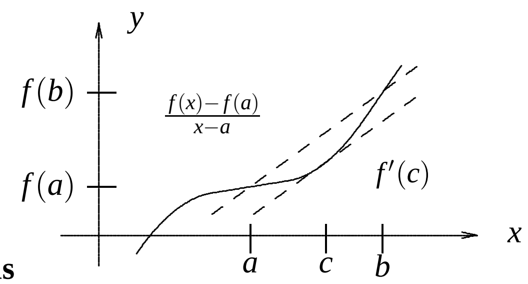
If we assume that the function f is differentiable at c, then the two last limits exist and are equal to $f'(c)$. Therefore we both have $f'(c) \geq 0$ and $f'(c) \leq 0$, which implies $f'(c) = 0$. If on the other hand the function f has a local maximum at c, then the signs in the limits are reversed but the conclusion is the same, namely $f'(c) = 0$ and we have:

If $f: I \to \mathbb{R}$ is a function on an open interval I which has a local extremum at the point c and f is differentiable at c, then $f'(c) = 0$.

The mean value theorem

To further investigate the sign of the derivative we need the *mean value theorem* which states that if f is a continuous function on the closed interval $[a,b]$ and differentiable on the open interval (a,b) then there is $c \in (a,b)$ such that

$$f(b) - f(a) = f'(c)(b-a).$$



Monotone intervals
Suppose the function f is continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b).

(i) If $f'(x) \geq 0$ for all $x \in]a,b[$, then f is increasing on $[a,b]$.

(ii) If $f'(x) \leq 0$ for all $x \in]a,b[$, then f is decreasing on $[a,b]$.

(iii) If $f'(x) > 0$ for all $x \in]a,b[$, then f is strctly increasing on $[a,b]$.

(iv) If $f'(x) < 0$ for all $x \in]a,b[$, then f is strictly decreasaing $[a,b]$.

Monotone intervals and extrema
Let f be a continuous function on the closed interval $[a,b]$, let $a \leq a_1 < b_1 \leq b$ and assume that f is differentiable at all points in the open interval (a_1, b_1) except possibly at the point c.

(i) If $f'(x) \geq 0$ for all $x \in (a_1, c)$, and $f'(x) \leq 0$ for all $x \in (c, b_1)$, then f has a local maximum at c.

(ii) If $f'(x) \leq 0$ for all $x \in (a_1, c)$, and $f'(x) \geq 0$ for all $x \in (c, b_1)$, then f has a local minimum at c.

Example 9.4.1
Find the intervals of monotonicity and local extrema of the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) := x^3 - x^2 - 5x$.
Solution: We start by finding the derivative of f, $f'(x) = 3x^2 - 2x - 5$. We then use the solution formula for quadratic equation to find it's zeros.

$$x = \frac{2 \pm \sqrt{4 - 4 \cdot 3(-5)}}{3 \cdot 2} = \frac{1}{3} \pm \frac{\sqrt{64}}{6} = \frac{1 \pm 4}{3}, \quad \text{i.e. } x_1 = \frac{5}{3},\ x_2 = -1.$$

The derivative is continuous so between zeros the derivative has the same sign. We then find the sign of the derivative of each of the intervals between the zeros. We find that on the interval $]-\infty, -1[$ the derivative is positive, it is negative on $]-1, \frac{5}{3}[$ and positive on á $]\frac{5}{3}, +\infty[$. Then we can conclude that f is increasing on $]-\infty, -1[$, decreasing on $]-1, \frac{5}{3}[$ and increasing on $]\frac{5}{3}, +\infty[$ and that f has local maximum in $x = -1$ and local minimum in $x = \frac{5}{3}$.

33 Increasing and decreasing functions; derivative tests

33.1 Plotting graphs of a function

33.1.1 Handout

The following gives a summary of how information on the derivative of a function is used to plot its graph:

(1) First find the domain of the function, including the location of any poles.

(2) Find the function values or limits of the function at domain endpoints, in particular the limits of the function as $x \to \infty$ and $x \to -\infty$ if the domain is unlimited. If the funciton has a pole at $x = a$ then one needs to investigate both $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$.

(3) Find the intersections of the graph with the x- and y-axes, i.e. on the one hand solve the equation $f(x) = 0$ and on the other hand compute the value $f(0)$.

(4) Compute the derivative of the function and find all points where the derivative is zero or undefined, i.e. solve the equation $f'(x) = 0$ and find all x in the domain such that $f'(x)$ is not defined.

(5) Split the domain into intervals where the function is monotonic and well defined and find the sign of the derivative in each such interval. Recall that the function is monotonic in an interval unless: (a) the derivative has a zero within the interval, (b) the derivative is not defined everywhere in the interval, or (c) the function is discontinuous within the interval.

(6) Use this information to find the locations of all local maxima and minima.

(7) Compute and plot the function values at the maxima and minima.

(8) Compute and plot function values at a handful of intermediate points.

Example 9.5.1
Find information sufficient to draw a reasonable graph of $f : \mathbb{R} \to \mathbb{R}$; $f(x) := x^3 - x^2 - 5x$. **Solution:** Most of what we have to do has already been done in example 9.4.1. We can see the function is continuous everywhere. We check the limits $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to-\infty} = -\infty$. Intersection with $y-$axis is at $(0,0)$ and zeros are in 0 and the zeros of $x^2 - x - 5$ which are

$$x = \frac{1 \pm \sqrt{1-4(-5)}}{2} = \frac{1 \pm \sqrt{21}}{2}.$$

We know all intervals of monotonicity from example 9.4.1 and where f has extrema. We have to find the values of f in -1 and $\frac{5}{3}$, they are $f(-1) = 3$ and $f(5/3) = \frac{-175}{27}$. By finding a couple of other values where they are needed we can get enough information for a reasonable picture.

Example 9.5.2
Find information sufficient to draw a reasonable graph of $f(x) := \frac{x^3-2x^2}{\ln(x)}$.
Solution:

1. We start by finding the domain. ln is only defined for positive numbers and $\ln(1) = 0$ so f is not defined in $x = 0$.

2. We find the limits of f in $+\infty$ and 0. A third degree polynomial increases much faster than $\ln(x)$ when $x \to \infty$ so $\lim_{x\to\infty} f(x) = +\infty$. Then $\lim_{x\to 0^+} x^3 - 2x^2 = 0$ and $\lim_{x\to 0^+} \ln(x) = -\infty$ so $\lim_{x\to 0^+} f(x) = 0$. Now $\lim_{x\to 1} x^3 - 2x^2 = -1$. Then we get that $\lim_{x\to 1^+} f(x) = -\infty$ and $\lim_{x\to 1^-} f(x) = +\infty$ because $\lim_{x\to 1} \ln(x) = 0$ and $\ln(x)$ is positive for $x > 1$ and negative for $x < 1$.

3. We want to find all zeros of f. They are the same as for $x^3 - 2x^2$ on the positive real axis. $x^3 - 2x^2 = x^2(x-2)$ so the only zero is in $x = 2$.

4. Let's find the derivative of f.

$$f'(x) = \frac{(3x^2-4x)\ln(x) - (x^3-2x^2)\frac{1}{x}}{\ln(x)^2} = \frac{(3x^2-4x)\ln(x) - x^2+2x}{\ln(x)^2}$$

   We see that the derivative is defined on the whole domain of f. Then we find the zeros of $(3x^2-4x)\ln(x)-x^2+2x$ which turn out to be none. (We a skip the calculations required to show that.)

5. Now it's clear that the monotone intervals are $]0,1[$, where f is increasing, and $]1,+\infty[$ where f is also increasing.

6. There are no local extrema but to draw a reasonable graph it is good to calculate several values of f where needed, then we can draw a graph which gives a good grasp of most of the properties of f.

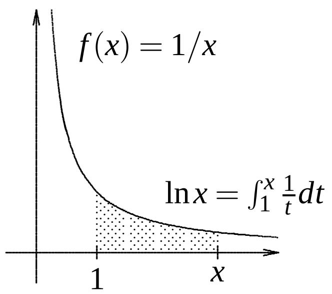
34 Natural logarithm and its derivative

34.1 Natural logarithm

34.1.1 Handout

The power function $x \mapsto x^m$ has the antiderivative $x \mapsto x^{m+1}/(m+1)$ for any rational number $m \neq -1$. The function $x \mapsto x^{-1} = 1/x$ is continuous on the positive real axis and therefore has an antideriviative. The particular antiderivative which takes on the value 0 at the point $x=1$ is called the *natural logarithm* and is given by the formula

$$\ln x = \int_1^x \frac{dt}{t}, \qquad x > 0.$$



For all real numbers x and y and all rational numbers r we have

$$\ln(xy) = \ln x + \ln y, \quad \ln(x/y) = \ln x - \ln y \quad \text{og} \quad \ln(x^r) = r \ln x.$$

The natural logarithm is strictly increasing and we have the limits

$$\lim_{x\to 0+} \ln x = -\infty \qquad \text{og} \qquad \lim_{x\to +\infty} \ln x = +\infty.$$

This implies that ln has an inverse, which we denote by exp. The inverse turns out to be an exponential function $\exp x = e^x$ with a number, e, as base. This number has the property that $\ln e = 1$. Any exponential function $x \mapsto a^x$ with base $a > 0$ can be written as $a^x = e^{(\ln a)x}$.
Example 10.8.1
Simplify $\ln(20) - 2\ln(2) + \ln(3)$.
Solution: By rules for logarithm we get

$$\ln(20) - 2\ln(2) + \ln(3) = \ln(20) - \ln(2^2) + \ln(3) = \ln(20\cdot 3) - \ln(4) = \ln\left(\frac{60}{4}\right) = \ln(15).$$

35 Derivative of inverse functions; exponential function and hyperbolic functions

36 Inverse trigonometric functions and their derivatives

37 l'Hôpital's rule

38 Indefinite integrals

38.1 Indefinite integrals

38.1.1 Handout

Let f be a function on an interval I. A function F is called the *antiderivative* or *primitive* of f if F is differentiable on the interval I and $F'(x) = f(x)$ for all $x \in I$. The antiderivative of a function f is often called the *indefinite integral* and is denoted by

$$F(x) = \int f(x)\,dx$$

note that if G is another antiderivative of f, then the difference satisfies the equation

$$(F-G)'(x) = F'(x) - G'(x) = f(x) - f(x) = 0$$

for every point $x \in I$. It follows that $F - G$ is a constant function and we therefore know that $G = F + C$, where C is some constant. In order to specify all possible intefinite integrals of the function $f(x)$, we always include the unspecified constant C and write

$$\int f(x)\,dx = F(x) + C.$$

Our knowledge of derivatives makes it possible to set up tables of antiderivatives. Such tables are very useful for computing integrals.

$f(x)$	$F(x) = \int f(t)dt$	$f(x)$	$F(x)$
x^n	$\frac{1}{n+1}x^{n+1} + C$	$\frac{1}{x}$	$\ln\lvert x\rvert + C$
e^x	$e^x + C$	a^x	$\frac{a^x}{\ln a} + C$
xe^x	$(x-1)e^x + C$	$\ln x$	$x\ln x - x + C$
$x^n e^s$	$x^n e^x - n\int x^{n-1}e^x dx$	$x^n \ln x$	$\frac{x^{n+1}}{n+1}\ln x - \frac{x^{n+1}}{(n+1)^2} + C$
$\sin x$	$-\cos x + C$	$\cos x$	$\sin x + C$
$\tan x$	$\ln\lvert\frac{1}{\cos x}\rvert + C$	$\cot x$	$\ln\lvert\sin x\rvert + C$
$\frac{1}{\sin x}$	$\ln\lvert\frac{1}{\sin x} - \cot x\rvert + C$	$\frac{1}{\cos x}$	$\ln\lvert\frac{1}{\cos x} + \tan x\rvert + C$
$\frac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\frac{x}{a} + C$	$\frac{dx}{a^2+x^2}$	$\frac{1}{a}\tan^{-1}\frac{x}{a} + C$
$\frac{1}{a^2-x^2}$	$\frac{1}{2a}\ln\left\lvert\frac{x+a}{x-a}\right\rvert + C$	$\sqrt{x^2 \pm a^2}$	$\frac{x}{2}\sqrt{x^2 \pm a^2} \pm \frac{a^2}{x}\ln\lvert x + \sqrt{x^2 \pm a^2}\rvert + C$
$\frac{1}{\sqrt{x^2 \pm a^2}}$	$\ln\lvert x + \sqrt{x^2 \pm a^2}\rvert + C$	$\sqrt{a^2 - x^2}$	$\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C$

39 Riemann sums and definite integrals

39.1 Definite integrals

39.1.1 Handout

Definite integral

To keep matters simple initially, consider a positive function on an interval, in other words let f be a function on an interval $[a,b]$ and assume that $f(x) \geq 0$ for all $x \in [a,b]$. We will want the "integral of the function f over the interval $[a,b]$" to mean the area of the region bounded by the x-axis, the graph of f, the vertical line $x = a$ and the vertical line $x = b$. This region in the plane is the set:

$$D = \{(x,y)\,;\, 0 \leq y \leq f(x), x \in [a,b]\}.$$

This task is easy if the function f is of a simple form, for example a constant, step function of if the graph is a combination of a finite number of line segments. To define the area of a general region D of this form requires approximating it from the inside and from the outside using polynomials, $A \subseteq D$ og $B \supseteq D$, and computing the areas of A and B. If there is an upper bound to the area of all polygons $A \subseteq D$ and if that is the same as the lower bound to the area of all polygons $B \supseteq D$, then that number $I \in \mathbb{R}$ is defined to be the *integral* of the function f over the area $[a,b]$ and the function f is said to be integrable over the interval $[a,b]$. The integral is denoted by

$$\int_a^b f(x)\,dx$$

and we view this as the area of the region D.

The formal definition of the integral needs to be a bit more general since it is not limited to positive functions. Further, in the definition we only consider polygons which are unions of rectangles with sides parallel to the axes of the coordinate system.

A set $P = \{x_0, x_1, \ldots, x_n\}$ where $a = x_0 < x_1 < \cdots < x_n = b$ is a *partition* of the interval $[a,b]$. Consider a function $f : [a,b] \to \mathbb{R}$ and assume that the function f is *bounded*, so that there exist constants m and M, such that

$$m \leq f(x) \leq M, \qquad x \in [a,b].$$

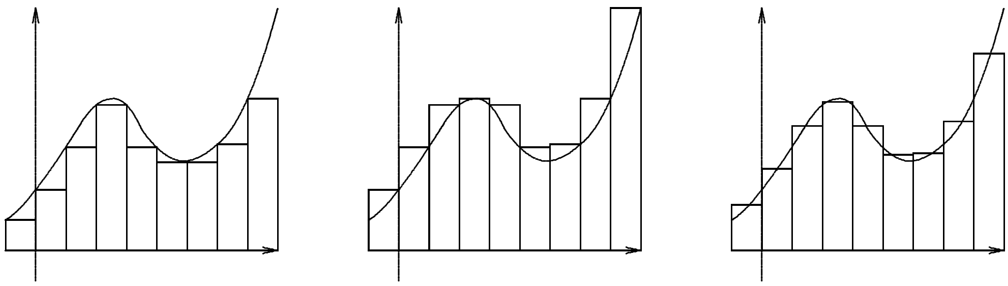
Sums of the form

$$U = \sum_{i=1}^{n} m_i(x_i - x_{i-1}) \qquad \text{og} \qquad Y = \sum_{i=1}^{n} M_i(x_i - x_{i-1}),$$

where $m_i \leq f(x) \leq M_i$ for all $x \in [x_{i-1}, x_i]$, are called *lower sums* and *upper sums for the function* f and a sum of the form

$$S = \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1})$$

where $t_i \in [x_{i-1}, x_i]$ is *Riemann-sum for the function* f. We have the obvious inequality $U \leq S \leq Y$ for all lower, Riemann and upper sums U, S and Y.



Lower sum. Upper sum. Riemann sum.

The bounded function $f : [a,b] \to \mathbb{R}$ is said to be *integrable* if for every $\varepsilon > 0$ there is a lower sum U and upper sum Y for f such that $Y - U < \varepsilon$.

If f is integrable then one can find an increasing sequence of lower sums $(U_j)_{j=1}^{\infty}$ and decreasing sequence of upper sums $(Y_j)_{j=1}^{\infty}$ such that $Y_j - U_j \to 0$. In this case both sequences converge to the same limit which is defined to be the *integral of the function f over the interval* $[a,b]$ denoted by

$$\int_a^b f(x)\,dx.$$

It should be noted that x in this case is *not* a variable, i.e. $\int_a^b f(x)\,dx$ is not a function of x. Once the integral has been evaluated there is no x any more in the formula. For this reason the term to "integrate out x" is commonly used.

In this definition it is assumed that $a \leq b$. If $a = b$, then the integral is 0. If $a > b$, we define

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$$

Example 10.1.1

Find upper sum and lower sum with points of division in the integers for $f := x^2 + 4x$ on the interval $[0,8]$.

Solution: We see the function is strictly increasing so by choosing the value taken at the right end on each interval we get an upper sum and by choosing the value at the left end we get a lower sum. Every interval is of length 1 so we get, where Y denotes upper sum

$$Y = \sum_{k=1}^{8} f(k) = \sum_{k=1}^{8} k^2 + 4k = \sum_{k=1}^{8} k^2 + 4\sum_{k=1}^{8} k = 204 + 4 \cdot 36 = 348$$

and the lower sum U is

$$U = \sum_{k=1}^{8} f(k-1) = \sum_{k=0}^{7} k^2 + 4k = Y - f(8) = 252.$$

Computational formulas for integrals

(i) If f and g are two integrable functions on $[a,b]$, then the function $f+g$ is integrable and

$$\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

(ii) If k is a real number and f is integrable on $[a,b]$, then kf is integrable and

$$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx.$$

(iii) If $a < c < b$ and f is integrable on $[a,b]$, then f is integrable on bothe intervals $[a,c]$ and $[c,b]$ with

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

(iv) If f is integrable on $[a+c, b+c]$, then

$$\int_a^b f(x+c)\,dx = \int_{a+c}^{b+c} f(x)\,dx.$$

Example 10.2.1
Calculate the integral

$$\int_0^1 x - k\,dx + \int_2^3 (x-1) + k\,dx$$

Solution: We use known rules for integration and get

$$\begin{aligned}
&\int_0^1 x - k\,dx + \int_2^3 (x-1) + k\,dx \\
&= \int_0^1 x - k\,dx + \int_1^2 x + k\,dx \\
&= \int_0^1 x\,dx - \int_0^1 k\,dx + \int_1^2 x\,dx + \int_1^2 k\,dx \\
&= \int_0^2 x\,dx - k + k = \frac{2^2}{2} - \frac{0^2}{2} = 2.
\end{aligned}$$

Monotonic functions are integrable

Let $f : [a,b] \to \mathbb{R}$ be an increasing function. Then f is bounded since $f(a) \le f(x) \le f(b)$ for all $x \in [a,b]$. Let $P = \{x_0, \ldots, x_n\}$ be an *even* division of $[a,b]$, i.e. $x_i = a + i(b-a)/n$. Then $x_i - x_{i-1} = (b-a)/n$. We can now set $m_i = f(x_{i-1})$ and $M_i = f(x_i)$. Since f is increasing the numbers $U = \sum_{i=1}^n m_i(b-a)/n$ and $Y = \sum_{i=1}^n M_i(b-a)$ are lower and upper sums for the function f and we have

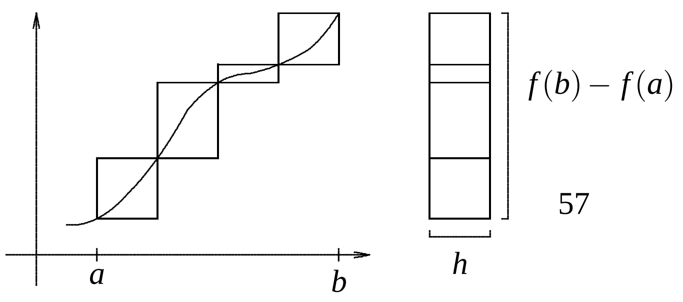
$$Y - U = \sum_{i=1}^n (M_i - m_i)(b-a)/n = \sum_{i=1}^n (f(x_i) - f(x_{i-1}))(b-a)/n = (f(b) - f(a))(b-a)/n.$$

Note that we have used the following fact:

$$\begin{aligned}
&\sum_{i=1}^n (f(x_i) - f(x_{i-1})) \\
&\qquad = f(x_1) - f(x_0) + f(x_2) - f(x_1) + f(x_3) - f(x_2) + \cdots + f(x_n) - f(x_{n-1}) \\
&\qquad = f(x_n) - f(x_0) = f(b) - f(a).
\end{aligned}$$

It follows that by choosing the division fine enough, i.e. choosing n large enough, we can make the difference $Y - U$ as small as we like. We have thus shown that an increasing function is integrable.

Note that since the function is monotonic the rectangles defined by the points $(x_i, f(x_i))$ og $(x_{i+1}, f(x_{i+1}))$ do not overlap. We can therefore think of them as stacked on top of each other to form a rectangle



with area $h \cdot (f(b) - f(a))$. The area of this rectangle is equal to $Y - U$. But by letting h tend to 0 we see that the area – and thus $Y - U$ – can be made abitrarily small and therefore the function is integrable.

If f is decreasing, then $-f$ is increasing so $-f$ is integrable. Rule (ii) in the preceding section using $k = -1$ then shows that f is integrable. From this we obtain the general result: *A monotonic function on* $[a, b]$ *is integrable.*

Continuous functions are intregrable

Consider a continuous function $f : [a, b] \to \mathbb{R}$ and select an evenly spaced partition as previously done. Since the function f is continuous it takes a maximum and a minimum in some points c_i and d_i within each subdivision $[x_{i-1}, x_i]$. This implies that $f(c_i) = \min\{f(x)\,; x \in [x_{i-1}, x_i]\}$ and $f(d_i) = \max\{f(x)\,; x \in [x_{i-1}, x_i]\}$. The corresponding upper and lower sums therefore satisfy

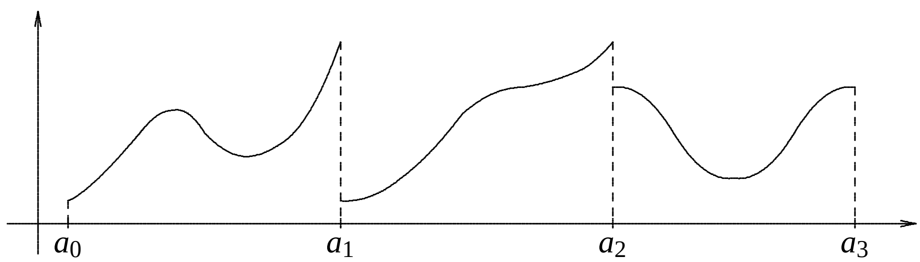
$$Y - U = \sum_{i=1}^{n} (M_i - m_i)(b - a)/n = \sum_{i=1}^{n} (f(d_i) - f(c_i))(b - a)/n$$

Given a number $\varepsilon > 0$, we can choose the partition fine enough so that $0 \leq f(d_i) - f(c_i) < \varepsilon/(b - a)$ for all n. The number of terms in the sum is n, so we have a bound on the difference: $Y - U < \varepsilon$. We have therefore shown that *any continuous real function defined on an interval* $[a, b]$ *is integrable.*

The last two results can be combined to the following: *If f is a bounded function on* $[a, b]$ *and if there is a partion of* $[a, b]$ *such that within each subdivision f is either monotonic or continuous, then f is integrable* and the integral is the sum of the integrals within the subdivisions, i.e.

$$\int_a^b f(x)\,dx = \sum_{i=1}^{n} \int_{a_{i-1}}^{a_i} f(x)\,dx$$

where a_i are the endpoints of the sub-intervals.



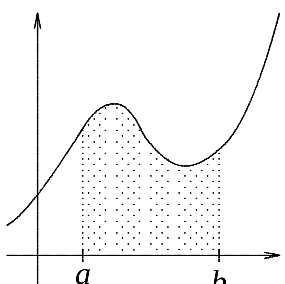
40 Fundamental theorem of calculus

40.1 Fundamental theorem of calculus

40.1.1 Handout

Fundamental theorem of calculus

$\int_a^b f(x)dx = F(b) - F(a)$



Integration and differentiation are the main operations of calculus. Their relationship is described by two theorems which in combination are called the fundamental theorem(s) of calculus:

(i) If the function f is continuous on $[a,b]$ and F is defined by

$$F(x) = \int_a^x f(t)\,dt, \qquad x \in [a,b]$$

then F is the antiderivative of f, i.e. $F'(x) = f(x)$ for all $x \in [a,b]$.

(ii) If f is a continuous function on $[a,b]$ and F is its antiderivative, then

$$\int_a^b f(t)\,dt = F(b) - F(a).$$

Example 10.6.1
Find the derivative of $F(x) := \int_0^x \sin^3(t^2)\,dt$.
Solution: By the fundamental theorem of calculus we get the derivative $F'(x) = \sin^3(x^2)$.

41 Rules of integration

41.1 Rules of integration

41.1.1 Handout

The second fundamental theorem of calculus states that the integral of a continuous function over an interval can be computed by finding an antiderivative and taking the difference of its values at the Two methode will be given to utilise this.
Integration by substitution
Let g be a continuous integrable function on the interval $[a,b]$ and f be a continuouos function on an interval I which contains the image $g([a,b])$. If the function F is an antiderivative for the continuous function f, then the chain rule gives

$$(F \circ g)'(x) = F'(g(x))g'(x) = f(g(x))g'(x)$$

and therefore $F \circ g$ is an antiderivative for $f(g(x))g'(x)$. The second fundamental theorem then gives

$$\int_a^b f(g(x))g'(x)\,dx = F(g(b)) - F(g(a)) = \int_{g(a)}^{g(b)} f(t)\,dt.$$

A common use of this method is to think of $u = g(x)$ as a new variable in the leftmost integral and this is inserted into the integral in place of $g(x)$ and then du must be inserted in place of $g'(x)\,dx$.
Using indefinite integrals this rule becomes

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du = F(u) + C = F(g(x)) + C,$$

where C is a constant.
Integration by parts
Consider two differentiable functions f and g with continuous derivatives. The Leibniz-rule states $(fg)' = f'g + fg'$ and from it follows

$$\begin{aligned}\int_a^b f(x)g'(x)\,dx &= \Big[f(x)g(x)\Big]_a^b - \int_a^b f'(x)g(x)\,dx \\ &= f(b)g(b) - f(a)g(a) - \int_a^b f'(x)g(x)\,dx.\end{aligned}$$

Writing this result in the form of indefinite integrals the rule becomes

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx.$$

Partial fractions expansion

Many rational functions can initially seem intractable for integration. In this case *partial fraction expansions* may be used to simplify the task, as described below.

Any polynomial q with real coefficients can be written in the form

$$q(x) = a(x-b_1)^{i_1}(x-b_2)^{i_2}\dots(x-b_n)^{i_n}(x^2+c_1x+d_1)^{j_1}(x^2+c_2x+d_2)^{j_2}\dots(x^2+c_mx+d_m)^{j_m}.$$

where a, b_n, c_n and d_n are real numbers, i_n are j_n natural numbers and the second order terms do not have real roots. It can be shown that if f is a rational function, $f(x) = p(x)/q(x)$, where p and q are polynomials with real coefficients, then f can be written as a sum of terms of the form

$$\frac{A}{(x-b_r)^k}$$

and the form

$$\frac{Bx+C}{(x^2+c_s+d_s)^l}$$

where A, B and C are real numbers, $k \le i_r$ and $l \le j_s$. Rational functions of this form are called *partial fractions* and are fairly easily integrated.

Now let $f(x) = p(x)/q(x)$ where p and q are polynomials. If p is of order greater than or equal to the order of q then we can write $f(x) = a(x) + b(x)/q(x)$ where a and b are polynomials and the order of b is lower than the order of q. Since polynomials (such as a) are easily integrated, we will assume p to be of a lower order than q.

To expand $f(x) = p(x)/q(x)$ into partial fractions one first factors $q(x)$ into a product of first and second order polynomials with real coefficients. Next define the equation

$$f(x) = \sum_{k=1}^{n}\sum_{l=1}^{i_k}\frac{A_{kl}}{(x-b_k)^{i_l}} + \sum_{k=1}^{m}\sum_{l=1}^{j_k}\frac{B_{kl}x+C_{kl}}{(x^2+c_kx+d_k)^{j_l}},$$

where the left-hand side, $f(x)$, is the ratio $p(x)/q(x)$ and the right hand side has as denominators all the terms forming $q(x)$.

The next step is to find the lowest common denominators of the right-hand side, simplify it and compare the coefficients. By solving the resulting equations one can find the missing coefficients A_i, B_i and C_i.

As an example, consider the rational function

$$f(x) = \frac{p(x)}{q(x)} = \frac{x^5}{x^4+3x^3+5x^2+5x+2}.$$

The order of p is higher than the order of q so we undertake polynomial division and rewrite $f(x)$ as

$$f(x) = x-3+\frac{4x^3+10x^2+13x+6}{x^4+3x^3+5x^2+5x+2}.$$

In this new setup we have the same $q(x)$, and by observing that $x = 1$ is a double root we can write

$$q(x) = x^4 + 3x^3 + 5x^2 + 5x + 2 = (x+1)^2(x^2 + x + 2)$$

where $x^2 + x + 2$ does not have a real root so we will solve the equation

$$\frac{4x^3 + 10x^2 + 13x + 6}{x^4 + 3x^3 + 5x^2 + 5x + 2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{Cx + D}{x^2 + x + 2}.$$

Finding a common denominator on the right hand side and comparing coefficients we see that we must have

$$\begin{aligned} A + C &= 4, \\ 2A + B + 2C + D &= 10, \\ 3A + B + C + 2D &= 13, \\ 2A + 2B + D &= 6. \end{aligned}$$

Solving this system of equations gives us $A = 9/4, B = -1/2, C = 7/4$ and $D = 5/2 = 10/4$ so the partial fractions become

$$f(x) = x - 3 + \frac{9}{4(x+1)} - \frac{1}{2(x+1)^2} + \frac{7x + 10}{4(x^2 + x + 2)}.$$

Example 10.7.1
Calculate the integral $\int_0^{\pi/2} \sin(x)\cos(x)\,dx$.
Solution: We use the substitution $u = \sin(x)$, that gives $\frac{du}{dx} = \cos(x)$ and we get

$$\int_0^{\pi/2} \sin(x)\cos(x)\,dx = \int_{\sin(0)}^{\sin(\pi/2)} \frac{u\cos(x)}{\cos(x)}\,du = \int_0^1 u\,du = \frac{1^2}{2} - \frac{0^2}{2} = \frac{1}{2}.$$

Example 10.7.2
Calculate the integral $\int_0^{\pi} x^2 \sin(x)\,dx$.
Solution: We use integration by parts and get

$$\int_0^{\pi} x^2 \sin(x)\,dx = [-x^2\cos(x)]_0^{\pi} - \int_0^{\pi} -2x\cos(x)\,dx$$

We calculate the value in brackets (note that $\sin(0) = \sin(\pi) = 0$) and use integration by parts again

$$= \pi^2 - [-2x\sin(x)]_0^{\pi} + \int_0^{\pi} -2\sin(x)\,dx = \pi^2 - 2[-\cos(x)]_0^{\pi} = \pi^2 - 2(1+1) = \pi^2 - 4.$$

Example 10.7.3
Calculate the integral

$$\int_3^5 \frac{x^4 + 2x^2 - 4x - 1}{x^3 - 3x^2 + 2x}\,dx.$$

Solution: We start by dividing with residue and get

$$\frac{x^4 + 2x^2 - 4x - 1}{x^3 - 3x^2 + 2x} = x + 3 + \frac{9x^2 - 10x - 1}{x^3 - 3x^2 + 2x} = x + 3 + \frac{9x^2 - 10x - 1}{x(x-1)(x-2)}..$$

Then we find partial fractions:

$$\frac{A}{x}+\frac{B}{(x-1)}+\frac{C}{x-2}=\frac{9x^2-10x-1}{x(x-1)(x-2)}$$

which gives us

$$\begin{aligned}&\frac{A(x-1)(x-2)+Bx(x-2)+Cx(x-1)}{x(x-1)(x-2)}\\&=\frac{Ax^2-3Ax-2A+Bx^2-2Bx+Cx^2-Cx}{x(x-1)(x-2)}\\&=\frac{9x^2-10x-1}{x(x-1)(x-2)}.\end{aligned}$$

and we get the equation system

$$\begin{aligned}A+B+C&=9,\\-3A-2B-C&=-10,\\2A&=-1.\end{aligned}$$

To solve this system we start by using the last equation to find $A=-\frac{1}{2}$. We add together the first two equations and get $-2A-B=-1$ and by putting in the known value of A we get $B=-2A-1=1+1=2$. Then the first equation gives $C=9+\frac{1}{2}-2=\frac{15}{2}$. Then we can integrate:

$$\begin{aligned}&\int_3^5\frac{x^4+2x^2-4x-1}{x^3-3x^2+2x}\,dx\\&=\int_3^5 x+3-\frac{1}{2x}+\frac{2}{x-1}+\frac{15}{2(x-2)}\,dx\\&=\left[\frac{x^2}{2}+3x-\frac{1}{2}\ln(x)+\ln(x-1)+\frac{15}{2}\ln(x-2)\right]_3^5\\&=\frac{5^2}{2}+3\cdot5-\frac{1}{2}\ln(5)+2\ln(4)+\frac{15}{2}\ln(3)-\frac{3^2}{2}-3\cdot3+\frac{1}{2}\ln(3)-2\ln(2)-\frac{15}{2}\ln(1)\\&=14-\frac{1}{2}\ln(5)+\ln(4)+8\ln(3).\end{aligned}$$

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50 Sequences

50.1 Sequences

50.1.1 Handout

Sequences

A function defined on $\mathbb{N}$ or $\mathbb{N}_0$ with values in some set is a *sequence*. A sequence is a *real sequence*, if the set is $\mathbb{R}$, but a *complex sequence*, if the set is $\mathbb{C}$. The function values, $a(n)$, are usually written a_n. Sequences a, b, $c, \ldots$, can be defined in different ways, e.g. using formulas such as

$$a_n = n, \qquad b_n = \frac{1}{n}, \qquad c_n = \frac{1}{2^n}, \qquad d_n = \frac{1}{n(n+1)}, \quad \ldots.$$

These same sequences can also be written as

$$(n)_{n=1}^{\infty}, \qquad (1/n)_{n=1}^{\infty}, \qquad (2^{-n})_{n=0}^{\infty}, \qquad \left(\frac{1}{n(n+1)}\right)_{n=1}^{\infty} \quad \ldots.$$

Sequences can be described by writing several terms, enough to clarify the rule being used. Thus sequences of even numberds, odd numbers, primes and negative powers of two can be described with

$$2,4,6,8,\ldots, \quad 1,3,5,7,\ldots, \quad 2,3,5,7,11,\ldots \quad \text{og} \quad \frac{1}{2},\frac{1}{4},\frac{1}{8},\frac{1}{16},\ldots.$$

Sequences can also be described using a *definition by induction*. in this case a few terms are written out in full, e.g. $a_1,\ldots,a_k$, followed by a formula which describes how a general a_n is computed based on the previous terms. A good example is the *Fibonacci-sequence*,

$$a_1 = 1,\ a_2 = 1,\ a_n = a_{n-2} + a_{n-1}, \quad n \geq 2.$$

This results in the sequence

$$1,1,2,3,5,8,13,21,\ldots.$$

Example 7.1.1
a) Find a formula for the $n-$th term of the sequence given by $1,2,4,8,16,\ldots$.
Solution: Each term is double the next term before, where the first term is $1=2^0$. Then the $n-$th term is 2^{n-1}.
b) Define the sequence from a) by induction.
Solution: We have that $a_1 = 1$ and every term thereafter is double the previous term which can by formulated by $a_n = 2a_{n-1},\ n \geq 2$.

51 Series

51.1 Series

51.1.1 Handout

Series

If $(a_n)_{n=1}^{\infty}$ is a sequence then we can form a new sequence $(s_n)_{n=1}^{\infty}$ by forming sums,

$$s_1 = a_1,\ \ s_2 = a_1 + a_2,\ \ s_n = a_1 + \cdots + a_n.$$

The general k'th term in this new sequence is the k-th *partial sum* of the sequence $(a_n)_{n=1}^{\infty}$. A *series* $\sum_{k=1}^{\infty} a_k$ is the sequence which consists of the partial sums of a given series $(a_k)_{k=1}^{\infty}$. Thus $\sum_{k=1}^{\infty} 1/k$ is the sequence

$$1,\ 1+\frac{1}{2},\ 1+\frac{1}{2}+\frac{1}{3},\ 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4},\ldots.$$

Example 7.2.1
Find a formula for the n-th term of the series $\sum_{k=1}^{+\infty} k$.
Solution: Here $\sum_{k=1}^{+\infty} k$ denotes the sequence where the $n-$th term is the $n-$th partial sum of $(k)_{k=1}^{+\infty}$, that is $s_n = \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$.

Example 7.2.2
Find the partial sum sequence of the sequence $(a_k)_{k=1}^{+\infty}$, which is given by $a_n := (-1)^n$.
Solution: We are to calculate the sum $\sum_{k=1}^{n}(-1)^k$. For $n=1$ the sum is -1. For $n=2$ it is 0 and then repeats,

$$\sum_{k=1}^{n}(-1)^k = \begin{cases} -1 \text{ if } n \text{ is odd} \\ 0 \text{ if } n \text{ is even} \end{cases}.$$

Arithmetic sequence

A sequence $(a_n)_{n=1}^{\infty}$ is an *arithmetic sequence* if the difference between adjacent terms is a fixed number m, so that $a_{k+1} - a_k = m$ for all k. We obtain a general description with

$$a_2 = a_1 + m,\ a_3 = a_2 + m = a_1 + 2m, \ldots,\ a_n = a_1 + (n-1)m.$$

In this case it is easy to find a formula form the n-th partial sum, as we can use the formula

$\sum_{k=1}^{n} k = \frac{1}{2}n(n+1)$ (easily proven by induction):

$$\begin{aligned} s_n &= \sum_{k=1}^{n}(a_1+(k-1)m) = na_1 + m\sum_{k=1}^{n}(k-1) \\ &= na_1 + m\sum_{k=1}^{n-1} k = na_1 + m\cdot\tfrac{1}{2}(n-1)n \\ &= n\cdot\frac{2a_1+(n-1)m}{2} = n\cdot\frac{a_1+a_n}{2}. \end{aligned}$$

From the last part of this equation we see that the partial sum in an arithmetic sequence is equal to the number of terms times the average of the first and last term.
Note that $(k)_{k=1}^{\infty}$ is an arithmetic sequence with $a_1 = 1$ and $m = 1$.

Example 7.3.1
In a arithmetic sequence $(a_n)_{n=1}^{\infty}$ the first term is 2 and the fifth term is 14. Find the sum of the first 100 terms.

Solution: The difference between first and fifth term is 12 so the difference of adjacent terms is 3. Then we calculate term 100 by $a_{100} = 2+(100-1)\cdot 3 = 299$. Then we use a formula for partial sum of arithmetic series and find

$$\sum_{k=1}^{100} a_k = 100\frac{a_1+a_{100}}{2} = 50(2+299) = 15050.$$

Geometric sequence

A *geometric sequence* or a *geometric progression* is a sequence $(a_k)_{k=1}^{\infty}$ such that the ratio between adjacent terms is a constant, i.e. there is a number $q \neq 0$, so that $a_{n+1}/a_n = q$ for all n. The number q is the *ratio* of the sequence.
Consider some examples of geometric sequences

$$1,-1,1,-1,1,-1,\ldots,\quad 1,\frac{1}{2},\frac{1}{4},\frac{1}{8},\ldots,\quad i,-1,-i,1,i,-1,\ldots.$$

The terms in a geometric progression are of the form

$$a_2 = a_1q,\ a_3 = a_2q = a_1q^2,\ \ldots,\ a_n = a_1q^{n-1}$$

and therefore the n-partial sum becomes

$$s_n = a_1(1+q+q^2+\cdots+q^{n-1}).$$

If $q=1$, then all the terms in the sequence (a_k) are the same and we obtain $s_n = a_1\cdot n$.
Now suppose $q \neq 1$ and multiply s_n by q. We then have

$$qs_n = a_1(q+q^2+\cdots+q^n) = s_n - a_1 + a_1q^n.$$

We can solve this equation for s_n to obtain

$$s_n = a_1\cdot\frac{1-q^n}{1-q}.$$

Example 7.4.1
The first term of a geometric sequence $(a_n)_{n=1}^{\infty}$ is 5 and the forth term is -40. Find the tenth term of the series $\sum_{n=1}^{\infty} a_n$.

Solution: We want to find the sum $s_{10} = \sum_{n=1}^{10} a_n$. We start by finding the quotient q, we know that $5q^3 = -40$ an thus $q = \sqrt[3]{-8} = -2$. Then we throw what we have into a formula:

$$s_{10} = \sum_{k=1}^{10} 5(-2)^{k-1} = a_1 \frac{1-q^{10}}{1-q} = 5\frac{1-1024}{3} = -1705.$$

Limits of sequences and series

A real sequence $(a_k)_{k=1}^{\infty}$ is said to be *convergent* with limit L if for every open interval I containing L there is $N \in \mathbb{N}$ so that $a_k \in I$ if $k \geq N$. We indicate this by writing

$$L = \lim_{k\to\infty} a_k.$$

Recall that any open interval containing L also includes an open interval of the form $(L-\varepsilon, L+\varepsilon) = \{x \in \mathbb{R};\, |x-L| < \varepsilon\}$. This implies that one can re-word the definition of the limit by stating that a sequence is convergent with limit L if for every $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that

$$|a_k - L| < \varepsilon \qquad \text{for all } k \geq N.$$

A corresponding definition for complex sequences exchanges the words "every open interval" by "every open disc".

An arithmetic sequence $a_k = a_1 + (k-1)m$ has a limit only if it is a constant seqence, i.e. $m = 0$, in which case the limit is a_1. A geometric sequence $a_k = a_1 q^{(k-1)}$ has a limit if and only if $|q| < 1$ or $q = 1$ with limit 0 if $|q| < 1$ but a_1 if $|q| = 1$.

The series $\sum_{k=1}^{\infty} a_k$ is said to be convergent if the sequence of partial sums is convergent and we then write the limit as $\sum_{k=1}^{\infty} a_k$.

It is easy to see that if a series is convergent then the terms must converge to 0. This statement can not be reversed as it is easy to find sequences which are not convergent but have terms which go to 0.

Earlier we computed the partial sums of a geometric progression as $a_k = a_1 q^{k-1}$, $a_1 \neq 0$. It is convergent only if $|q| < 1$ and

$$\sum_{k=1}^{\infty} a_1 q^{k-1} = a_1 \cdot \frac{1}{1-q}.$$

Example 7.5.1

Let $(a_n)_{n=1}^{\infty}$ be a geometric sequence with $a_1 := 5$ and $a_2 := 4$. Find the sum of the series $\sum_{k=1}^{\infty} a_k$.

Solution: We can see that the quotient is $q = \frac{4}{5} < 1$ and we put this into a formula:

$$\sum_{k=1}^{\infty} a_k = a_1 \frac{1}{1-q} = \frac{5}{1/5} = 25.$$

Example 7.5.2

Find the limit of the sequence $(a_n)_{n=1}^{\infty}$ with $a_n := \frac{n^2-1}{n^2+n+1}$.

Solution: We divide over and under bar with n^2:

$$\lim_{n\to\infty} \frac{n^2-1}{n^2+n+1} = \lim_{n\to\infty} \frac{1-1/n^2}{1+1/n+1/n^2} = \frac{\lim_{n\to\infty} 1-1/n^2}{\lim_{n\to\infty} 1+1/n+1/n^2} = \frac{1}{1} = 1.$$

Example 7.5.3
find whether the sequence $a_n := \cos(n\pi)$ is convergent.
Solution: Notice that $\cos(n\pi) = (-1)^{n-1}$, but that means that the sequence cannot be convergent as it takes two different values, 1 and -1, infinitely often.

52 Complex numbers

52.1 Complex numbers

52.1.1 Handout

Limitations of the real numbers
The number systems $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$ have their limitations and the same applies to the real numbers, $\mathbb{R}$. In the set of natural numbers subtraction is incomplete. In the set of integers division is incomplete. The rational numbers can not be used to describe the lengths of all line segments which occur in geometry.
The square of a positive number is always greater than or equal to zero and therefore there is no solution to the equation $x^2 = -1$. The same can be said for the general quadratic equation $ax^2 + bx + c = 0$ with $a \neq 0$. It has no solution among the real numbers if $D = b^2 - 4ac < 0$. Many other examples can be given of polynomials of even order with no roots in $\mathbb{R}$. Polynomials p of odd order always have a root.
It is a natural next question whether the real numbers system can be extended to a number system where one can solve the second order equation $x^2 = -1$ and whether such a number system gives solutions to other equations which can not be solved in $\mathbb{R}$.
Imagine that there is a number system which includes the real numbers as a subset and that there is an element i which satisfies $i^2 = -1$. The i is of course not a real number. Assume further that all the rules of arithmetic for reals still apply in the system. For example we then have $ia = ai$ for all real numbers a.
Now consider real numbers a, b, c and d. The rules above along with rules of arithmetic for real numbers give methods of adding and multiplying the numbers $a+ib$ and $c+id$ with

$$(a+ib)+(c+id) = a+(c+ib)+id = (a+c)+i(b+d)$$

and

$$\begin{aligned}(a+ib)(c+id) &= ac+ibc+aid+ibid \\ &= ac+ibc+iad+i^2bd \\ &= (ac-bd)+i(ad+bc).\end{aligned}$$

These two formulas give a recipe for how to add and multiply the numbers $a+ib$ and $c+id$ so as to obtain a number of the same form.
The complex number plane
Now let us revisit the question of whether and how one can extend $\mathbb{R}$ to a larger number system where there is a number i satisfying $i^2 = -1$. As it turns out such a system exists and every number in the system can be written in the form $a+ib$ where a and b are real numbers.
Consider the set of all vectors in a plane. Every vector has coordinates $(a,b) \in \mathbb{R}^2$ indicating where the endpoint is if the initial point is at the origin. The set of all vectors has the operations of addition and multiplication by a number. The addition is described by

$$(a,b)+(c,d) = (a+c, b+c).$$

Now, multiplication on $\mathbb{R}^2$ can be defined in the light of the formula described earlier. For the system to work we need something like $(a+ib)(c+id) = (ac-bd)+i(ad+bc)$ and therefore we simply define

$$(a,b)(c,d) = (ac-bd, ad+bc).$$

The number plane $\mathbb{R}^2$ with regular addition and this multiplication is called the system of *complex numbers* , denoted by $\mathbb{C}$. The next step is to establish all the usual rules of arithmetic:

$\big((a,b)+(c,d)\big)+(e,f) = (a,b)+\big((c,d)+(e,f)\big)$	*(associative law for addition),*
$\big((a,b)(c,d)\big)(e,f) = (a,b)\big((c,d)(e,f)\big)$	*(associative law for multiplication),*
$(a,b)+(c,d) = (c,d)+(a,b)$	*(commutative law for addition),*
$(a,b)(c,d) = (c,d)(a,b)$	*(commutative law for multiplication),*
$(a,b)\big((c,d)+(e,f)\big) = (a,b)(c,d)+(a,b)(e,f)$	*(distributive law),*
$(a,b)+(0,0) = (a,b)$	*($(0,0)$ is the additive unit),*
$(1,0)(a,b) = (a,b)$	*($(1,0)$ is the multiplicative unit).*

The number $(-a,-b)$ is the additive inverse of (a,b). Note that the equation $(a,b)(a,-b) = (a^2+b^2,0)$ implies that every number $(a,b) \neq (0,0)$ has a multiplicative inverse

$$\Big(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\Big),$$

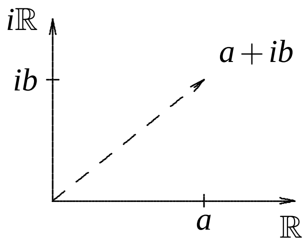
which implies that

$$(a,b)\Big(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\Big) = (1,0).$$

Now note that

$$(a,0)(c,d) = (ac,ad) = a(c,d).$$

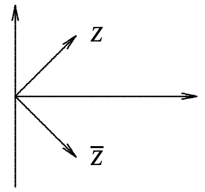
which implies that multiplication by the vector $(a,0)$ is the same as multiplication by the number a. Similarly the set of vectors of the form a and the vector $(a,0)$ and we simply consider the horizontal axis $\{(x,0) \in \mathbb{R}^2; x \in \mathbb{R}\}$ as the real line $\mathbb{R}$. In particular we simply write 1 in place of $(1,0)$ and 0 for $(0,0)$



The set $\mathbb{R}$ is called the *real axis* in the complex plane but the set $i\mathbb{R} = \{iy; y \in \mathbb{R}\}$ is the *imaginary axis*.
Consider the vector $(0,1)$. This vector has the property $(0,1)^2 = (-1,0)$. A special symbol is used to denote this vector: $i = (1,0)$. Every vector (a,b) can now be written as the composite $(a,b) = a(1,0)+b(0,1)$. Since we do not distinguish between the real number 1 and the complex number $(1,0)$ we have the representation

$$(a,b) = (a,0)(1,0)+(b,0)(0,1) = a+ib.$$

Real parts, imaginary parts and conjugates
If z is a complex number we can write $z = x+iy$ where x and y are real numbers. The number x is called the *real part* of the number z and the number y is the *imaginary part* . We denote the real part by $\operatorname{Re} z$ and the imaginary part by $\operatorname{Im} z$.



A complex number z *real* if $\mathrm{Im}\, z = 0$ and it is *pure imaginary* or simply *imaginary* if $\mathrm{Re}\, z = 0$.
If $z \in \mathbb{C}$ with $x = \mathrm{Re}\, z$ and $y = \mathrm{Im}\, z$, then $\bar{z} = x - iy$ is the *complex conjugate* or *conjugate* of the number z. Note that $\bar{z}$ is the reflextion of z about the real axis and $\bar{\bar{z}} = z$.
Some simple rules apply to the conjugate:

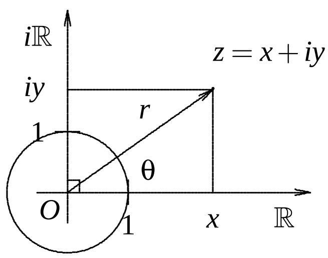
$$\begin{aligned} z\bar{z} &= (x+iy)(x-iy) = x^2 + y^2, \\ z + \bar{z} &= 2x = 2\,\mathrm{Re}\, z, \\ z - \bar{z} &= 2i\mathrm{Im}\, z, \\ \overline{zw} &= \bar{z}\bar{w}. \end{aligned}$$

The first equation gives a new way to write the multiplicative inverse:

$$z^{-1} = \frac{1}{z} = \frac{x - iy}{x^2 + y^2}, \qquad z \neq 0.$$

Note that z is real if and only if $z = \bar{z}$ and z is pure imaginary if and only if $z = -\bar{z}$.

Length and angular coordinate



If $z \in \mathbb{C}$, $x = \mathrm{Re}\, z$ and $y = \mathrm{Im}\, z$, the number

$$|z| = \sqrt{x^2 + y^2},$$

is the *length*, or *absolute value* of the complex number z. If $\theta \in \mathbb{R}$ and the complex number z can be written in the form

$$z = |z|(\cos\theta + i\sin\theta),$$

then the number θ is the *angular coordinate* of the complex number z. The trigonometric functions cos and sin are periodic with period 2π and therefore all numbers of the form $\theta + 2\pi k$ with $k \in \mathbb{Z}$ are also angular coordinates for z.
The ordered pair $(|z|, \theta)$ provides the *polar coordinates* of the number z.
Using the rules of arithmetic we can write the conjugate and length as follows:

$$\begin{aligned} |\bar{z}| &= |z|, \\ z\bar{z} &= |z|^2, \\ z^{-1} &= \frac{\bar{z}}{|z|^2}, \qquad z \neq 0. \end{aligned}$$

Powers

If z is a complex number we can define non-negative integer powers to obtain $z^0 = 1$, $z^1 = z$, and $z^n = z \cdots z$ where all the components are the same and their number is $n \geq 2$. The negative powers are defined first by taking z^{-1} as the multiplicative inverse of z and for negative n set $z^n = (z^{-1})^{|n|}$. This gives the same power rules as for real numbers:

$$\begin{aligned} z^n \cdot z^m &= z^{n+m}, \\ \frac{z^n}{z^m} &= z^{n-m}, \\ z^n \cdot w^n &= (zw)^n, \\ (z^n)^m &= z^{nm}. \end{aligned}$$

The unit circle
The unit circle $\mathbb{T}$ consists of all complex numbers with absolute value 1. Every z in $\mathbb{T}$ can therefore be written in the form $z = \cos\alpha + i\sin\alpha$. Take another such number, $w = \cos\beta + i\sin\beta$ and multiply the two to see what happens:

$$\begin{aligned} zw &= (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta) \\ &= (\cos\alpha\cos\beta - \sin\alpha\sin\beta) + i(\sin\alpha\cos\beta + \cos\alpha\sin\beta) \\ &= \cos(\alpha+\beta) + i\sin(\alpha+\beta). \end{aligned}$$

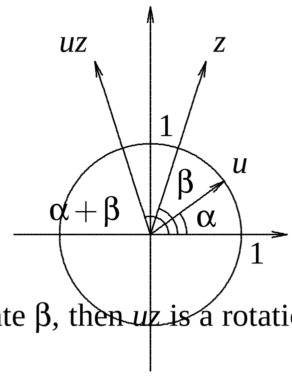
In this we have used the angle sum and difference rules. This formula provides a rule attributed to de Moivre:

$$\big(\cos\theta + i\sin\theta\big)^n = \cos(n\theta) + i\sin(n\theta).$$

Geometric interpretation of multiplication
Let z and w be two complex numbers of length $|z|$ and $|w|$ and angular coordinates α and β. We obtain

$$zw = |z||w|\big(\cos(\alpha+\beta) + i\sin(\alpha+\beta)\big).$$

wich tells us that the length of the product is the product of the lengths of z and w and that the angular coordinates of the product is the sum of the angular coordinates of z and w.



If $u \in \mathbb{T}$ is a number on the unit circle with angular coordinate β, then uz is a rotation of z by the angle β.
Triangular inequality
Consider two complex numbers z and w and let us do some computations:

$$\begin{aligned} |z+w|^2 &= (z+w)(\overline{z+w}) = (z+w)(\bar{z}+\bar{w}) \\ &= z\bar{z} + z\bar{w} + w\bar{z} + w\bar{w} \\ &= |z|^2 + z\bar{w} + \overline{z\bar{w}} + |w|^2 \\ &= |z|^2 + 2\mathrm{Re}\,(z\bar{w}) + |w|^2. \end{aligned}$$

Now note that

$$|\mathrm{Re}\,z| \leq |z| \qquad \text{and} \qquad |\mathrm{Im}\,z| \leq |z|.$$

The first inequality implies

$$|z+w|^2 \leq |z|^2 + 2|z||w| + |w|^2 = (|z|+|w|)^2.$$

Taking the square root on each side of the inequality signe gives the *triangular inequality*:

$$|z+w| \leq |z| + |w|.$$

Using this on the terms $z-w$ and w in place of z and w, gives $|z| = |(z-w)+w| \leq |z-w| + |w|$. Equivalently, $|z| - |w| = |(z-w)+w| \leq |z-w|$. If we switch the roles of z and w in the inequality we obtain $-|z| + |w| \leq |w-z| = |z-w|$. The last two inequalities are usually combined in a different version of the triangle inequality.

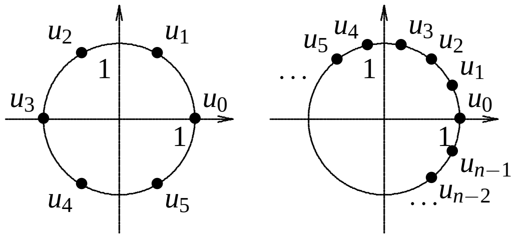
$$||z| - |w|| \leq |z-w|.$$

Unit roots
Consider the equation $z^n = 1$, where $n \geq 2$ is a natural number. If z solves the equation then $1 = |z^n| = |z|^n$ which implies $|z| = 1$ and we can write $z = \cos\theta + i\sin\theta$. The rule of de Moivres implies

$$\cos(n\theta) + i\sin(n\theta) = (\cos\theta + i\sin\theta)^n = z^n = 1.$$

The number 1 has the angular coordinate $2\pi k$ for any $k \in \mathbb{Z}$ and this number implies that $n\theta$ is an integer multiple of 2π so possible angular coordinates of the number z are given by

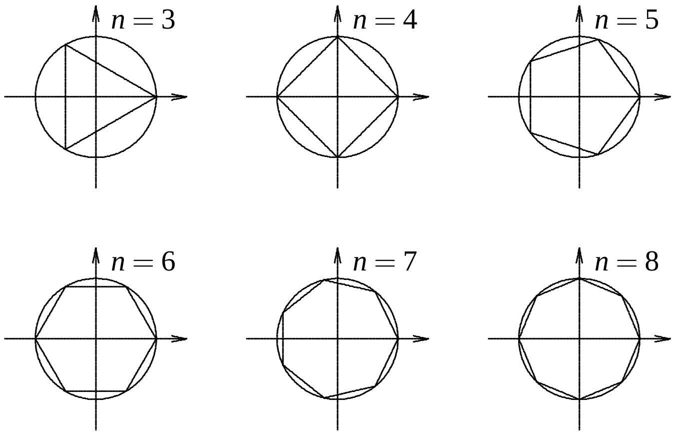
$$\theta = 2\pi k/n, \qquad k \in \mathbb{Z}.$$



If two integers k_1 and k_2 have the same remainder after integer dision by n, then $\cos(2\pi k_1/n) = \cos(2\pi k_2/n)$ and $\sin(2\pi k_1/n) = \sin(2\pi k_2/n)$. This tells us that the equation $z^n = 1$ has n distinct solutions $u_0, \dots, u_{n-1}$, called the *n-th roots of* 1 and these are given by the formula

$$u_k = \cos(2\pi k/n) + i\sin(2\pi k/n), \qquad k = 0, 1, 2, \dots, n-1.$$

These numbers are all on the unit circle. Note that $u_0 = 1$, $u_k = u_1^k$ for $k = 1, \dots, n-1$, and that these are placed at the vertices of a regular n-sided polygon, where we have defined a two-sided polygon to be the line seqment $[-1, 1]$.



Roots

Let $w = s(\cos\alpha + i\sin\alpha)$ be a given complex number of length $s \geq 0$ with angular coordinate α and let us seek a solution to the equation $z^n = w$. If z is such a solution and and u is an n-th unit root, then we have $(zu)^n = z^n u^n = z^n = w$ and hence zu is also a solution. Since there are n distinct unit roots this implies that once we find a single solution z_0 we find n different solutions $z_0 u$ by inserting all the n different unit roots for u. Now let z_0 be the complex number given by

$$z_0 = s^{\frac{1}{n}}\big(\cos(\alpha/n) + i\sin(\alpha/n)\big)$$

and raise this to the n-th power.

$$\begin{aligned} z_0^n &= \big(s^{\frac{1}{n}}\big)^n \big(\cos(\alpha/n) + i\sin(\alpha/n)\big)^n \\ &= s\big(\cos(n\alpha/n) + i\sin(n\alpha/n)\big) = w \end{aligned}$$

It follows that we have a formula for one solution. By using the formula for the n-th unit roots we obtain the set of all solutions to the equation $z^n = w = \rho(\cos\alpha + i\sin\alpha)$, with

$$z_k = \rho^{\frac{1}{n}}\big(\cos((\alpha + 2\pi k)/n) + i\sin((\alpha + 2\pi k)/n)\big), \qquad k = 0, \dots, n-1.$$

The formula can be described in words: The n roots are found by first finding a single solution z_0. It is then rotated by the angle $2\pi/n$ by multiplying with u_1 to obtain $z_1 = u_1 z_0$. Next z_1 is rotated by the angle $2\pi/n$ to $z_2 = u_1 z_1$, continuing in this manner until n different roots are found.

Example 6.1.1
Calculate $(2+5i)-(3-i)(5+2i)$.
Solution: As usually we first multiply and then add together.

$$(2+5i)-(3-i)(5+2i) = 2+5i-(15+6i-5i-i^2 2) = 2+5i-15-6i+5i-2 = -15+4i.$$

Example 6.1.2
Find real part, imaginary part, conjugate and multiplicative inverse of $(1+2i)$.
Solution: The real part is $\mathrm{Re}(1+2i) = 1$ and the imaginary part is $\mathrm{Im}(1+2i) = 2$. The conjugate is $\overline{1+2i} = 1-2i$. The multiplicative inverse is the conjugate divided by it's length squared.

$$\frac{1}{1+2i} = \frac{1-2i}{1^2+2^2} = \frac{1-2i}{5}.$$

Example 6.1.3
a) Find polar coordinates for $-1+i$.
Solution: The length is $\sqrt{(-1)^2+1^2} = \sqrt{2}$. The angle is $\frac{3\pi}{4}$, (we can see that by noting $\frac{3\pi}{4} = \frac{\pi}{2} + \arctan(1/1)$ and drawing a picture).
b) Calculate $(-1+i)^8$.
Solution: We found the polar coordinates in last part. We use them and apply De Moivre's rule:

$$(-1+i)^8 = (\sqrt{2}\cos(3\pi/4)+\sqrt{2}(3\pi/4))^8 = \sqrt{2}^8(\cos(8\cdot 3\pi/4)+\sin(8\cdot 3\pi/4)) = 16(\cos(6\pi)+i\sin(6\pi)) = 16$$

Example 6.1.4
a) Find all fifth roots of 1.
Solution: We know the length of all fifth roots of unity is one. Then we find all θ on the interval $[0,2\pi[$ such that $\cos(5\theta) = 1$, but these are the values of θ such that $5\theta = k2\pi$, $k \in \mathbb{Z}$, that is $0, \frac{2\pi}{5}, \frac{4\pi}{5}, \frac{6\pi}{5}, \frac{8\pi}{5}$. Then we have all five possible angular coordinates.
b) Find all fifth roots of $(16-16i)$.
Solution: We start by finding one fifth root. The angle of $(16-16i)$ is $\frac{7\pi}{4}$ and the length of the number is $16\sqrt{2}$. Then we get the fifth root $\sqrt[5]{16\sqrt{2}}(\cos(7\pi/4\cdot 5)+i\sin(7\pi/4\cdot 5)) = \sqrt{2}\left(\cos\left(\frac{7\pi}{20}\right)+i\sin\left(\frac{7\pi}{20}\right)\right)$. By multiplying this solution with fifth roots of unity we get all the fifth roots, they are

$$\sqrt{2}\left(\cos(7\pi/20+\frac{2k\pi}{5})+i\sin(7\pi/20+\frac{2k\pi}{5})\right), \quad k = 0,1,2,3,4.$$

Polynomials

Polynomials with complex coefficients

Polynomials with complex coefficients are defined as before, being expressions of the form

$$P(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0.$$

where $a_0, \ldots, a_n$ are complex numbers and z is a variable which can take on complex values. We can view P as a function defined on $\mathbb{C}$ taking values in $\mathbb{C}$. The order of the polynomial is, as before, the largest value of j such that $a_j \neq 0$.
A complex number α is said to be a *zero* or a *root* of the polynomial P if $P(\alpha) = 0$.

Root of a second degree polynomial

We want to solve the equation $az^2+bz+c=0$ assuming that the coefficients a, b and c are complex numbers and that $a \neq 0$. This is done exactly as in the real case but the result will become more general.

Start by dividing both sides by a to obtain an equivalent equation $z^2+Bz+C=0$, where $B=b/a$ and $C=c/a$. The next step is to look at the first two terms, z^2+Bz and write those as a square plus a constant. In other words we want to write it in the form $(z+\alpha)^2$. Now recall that $(z+\alpha)^2 = z^2+2\alpha z+\alpha^2$. Therefore

$$0 = z^2+Bz+C = (z+\frac{B}{2})^2 - \frac{B^2}{4}+C.$$

And this implies that the original equation is equivalent to

$$0 = (az^2+bz+c)/a = \left(z+\frac{b}{2a}\right)^2 - \frac{b^2}{4a^2}+\frac{c}{a}.$$

By subtracting the number $-b^2/(4a^2)+c/a$ from both sides we again obtain an equivalent equation

$$\left(z+\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2}-\frac{c}{a} = \frac{b^2-4ac}{4a^2}.$$

The complex number $D=b^2-4ac$ is called the *discriminant* of the equation. If $D \neq 0$, then D has two square roots. Let $\sqrt{D}$ denote one of them. The the other is $-\sqrt{D}$ and we obtain two distinct solutions

$$z_1 = \frac{-b+\sqrt{D}}{2a} \qquad \text{and} \qquad z_2 = \frac{-b-\sqrt{D}}{2a}.$$

If $D=0$, there is one solution

$$z = \frac{-b}{2a}.$$

In the special case when $D \in \mathbb{R}$ and $D<0$ then we can select a squate root $\sqrt{D} = i\sqrt{|D|}$ and the solutions becomes

$$z_1 = \frac{-b+i\sqrt{|D|}}{2a} \qquad \text{and} \qquad z_2 = \frac{-b-i\sqrt{|D|}}{2a}.$$

Recall that if α is a positive real, then $\sqrt{\alpha}$ denotes the positive number which satisfies $(\sqrt{\alpha})^2 = \alpha$ and of course $\sqrt{0}=0$. If $\alpha \neq 0$ is a complex number and α is not a positive real number, then $\sqrt{\alpha}$ has no standard meaning. We just know that two complex numbers β and γ exist which are square roots of α and they have the property that $\gamma = -\beta$.

The fundamental theorem of algebra

At the outset we defined the complex number in order to solve equations with no real solutions. The fundamental theorem of algebra states that any polynomial of order ≥ 1 with complex coefficients are zeroes in $\mathbb{C}$. From this point of view the extension of the number system from the real line to the complex plane is a success. The proof of the fundamental theorem demands considerable knowledge of calculus and is usually given in the second year of university studies.

We will take the fundamental theorem for granted and consider some important consequences of the theorem. Polynomial division works in the same manner for polynomials with complex coefficients as for real coefficients. If we take a polynomial P of order $m \geq 1$ and divide $(z-\alpha)$ into it, we obtain

$$P(z) = (z-\alpha)Q(z)+C,$$

where Q is a polynomial of order $m-1$ and C is a complex number. By inserting $z=\alpha$ into thie equation we see that $C=P(\alpha)$. This gives us a factorization methods which states that $(z-\alpha)$ divides $P(z)$ if and only if α is a zero of P.
Now suppose α is a zero of P and that the order is $m\geq 2$. Then Q is of order $m-1$ and according to the fundamental theorem Q has a zero, β. We factor $z-\beta$ out of Q and obtain $P(z)=(z-\alpha)(z-\beta)R(z)$ where R is a polynomial of order $m-2$. This can be continued until we end up with a perfect factorization of P into first order terms,

$$P(z)=A(z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_m)$$

where $\alpha_1,\ldots,\alpha_m$ is an enumeration of all the roots, possibly repeteted, and $A\neq 0$ is a complex number.

Polynomials with real coefficients

The real numbers are always viewed as a part of the complex numbers and therefore a polynomial with real coefficients is also a polynomial with complex coefficients. The fundamental theorem of algebra therefore applies to these polynomials. Now suppose $P(z)$ is a polynomial of order $m\geq 1$ with real coefficients $a_0,\ldots,a_m$ and that $\alpha\in\mathbb{C}$ is a root. Suppose further that α is not a real number. Using the computational rules for conjugates, noting in particular that $\bar{a}_j=a_j$, we obtain

$$0=P(\alpha)=\overline{P(\alpha)}=\overline{\sum_{k=0}^{m}a_k\alpha^k}=\sum_{k=0}^{m}\overline{a_k}\overline{\alpha^k}=\sum_{k=0}^{m}a_k(\overline{\alpha})^k=P(\bar{\alpha}).$$

We have therefore shown that $\bar{\alpha}$ is also a zero of P. We can there fore factor out $(z-\alpha)$ and $(z-\bar{\alpha})$ Next note that

$$(z-\alpha)(z-\bar{\alpha})=z^2-(\alpha+\bar{\alpha})z+\alpha\bar{\alpha}=z^2-2(\mathrm{Re}\,\alpha)z+|\alpha|^2,$$

i.e. the product is a quadratic polynomial with real coefficients. It follows that when we factor out the two complex roots we have in fact written $P(z)$ as a product of two polynomials with real coefficients:

$$P(z)=\left(z^2-2(\mathrm{Re}\,\alpha)z+|\alpha|^2\right)Q(z),$$

and from this we can draw the surprising conclusion that any polynomial with real coefficients can be written as a product of polynomials of order one or two, with real coefficients.

Expanding the exponential function

We have seen how the domain of polynomials can be extended from the real number axis $\mathbb{R}$ to be the entire complex plane $\mathbb{C}$. This can in fact be done for many functions which are defined on subsets of the real line, so they have a natural domain in $\mathbb{C}$.
The exponential function $\exp:\mathbb{R}\to\mathbb{R}$ is the inverse of the natural logarithm which is defined by the integral

$$\ln x=\int_1^x\frac{dt}{t},\qquad x>0.$$

The number e is defined by $e=\exp(1)$. We will now extend the domain of exp to become all of $\mathbb{C}$ with the formula

$$\exp(z)=e^x(\cos y+i\sin y),\qquad z=x+iy\in\mathbb{C},\quad x,y\in\mathbb{R}$$

Euler's equations

Take a pure imaginary number $i\theta$ where $\theta\in\mathbb{R}$ and insert this into the new definition of the exponential function to obtain $e^{i\theta}=(\cos\theta+i\sin\theta)\in\mathbb{T}$. This shows how the function $\theta\mapsto e^{i\theta}$ projects the real line onto the unit circle.

Now consider the following two equations

$$e^{i\theta} = \cos\theta + i\sin\theta,$$
$$e^{-i\theta} = \cos\theta - i\sin\theta.$$

By adding the left and right hand sides we see that $e^{i\theta} + e^{-i\theta} = 2\cos\theta$. Next take the difference between the two. Then we obtain $e^{i\theta} - e^{-i\theta} = 2i\sin\theta$. This gives us certain relationships between the exponential function and the trigonometric functions, which are termed *Euler's equations*,

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2},$$
$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

Additive theorem for the exponential function
Recall that the exponential function $\exp : \mathbb{R} \to \mathbb{R}$, $x \mapsto e^x$, satisfies the equality $e^{a+b} = e^a e^b$ for all real numbers a and b. This rule is the *addition theorem* for the exponential function. Now take two complex numbers $z = x + iy$ and $w = u + iv$ to see how the rule generalizes when the domain of the exponential function has been expanded to the entire complex plane $\mathbb{C}$:

$$\begin{aligned} e^z e^w &= e^x(\cos y + i\sin y)e^u(\cos v + i\sin v) \\ &= (e^x e^u)(\cos y + i\sin y)(\cos v + i\sin v) \\ &= e^{x+u}(\cos(y+v) + i\sin(y+v)) \\ &= e^{(x+u)+i(y+v)} = e^{(x+iy)+(u+iv)} = e^{z+w}. \end{aligned}$$

We thus see that the additive theorem for the exponential function is valid on the entire complex plane.

Example 6.2.1
Find all roots of the polynomial $x^2 - 2 + 5$.
Solution: Here we use the solution formula for quadratic equation directly, the solutions are

$$x = \frac{2 \pm \sqrt{4-20}}{2} = 1 \pm \sqrt{-4} = 1 \pm 2i.$$

Example 6.2.2
Solve the equation $ix^2 + 2x + 2 + 2i$.
Solution: We use solution formula for quadratic equation and get the solutions

$$\frac{-2 \pm \sqrt{4 - 8i + 8}}{2i} = \frac{-2 \pm \sqrt{12 - 8i}}{2i} = \frac{-2}{2i} \pm \sqrt{\frac{12-8i}{(2i)^2}} = i \pm \sqrt{2i - 3}.$$

Example 10.9.1
Calculate $e^{\ln(5) + \frac{i\pi}{3}}$.
Solution: We put this into formula and get

$$e^{\ln(5)+\frac{i\pi}{3}} = 5(\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right) = \frac{5 + 5\sqrt{3}i}{2}.$$

Example 10.9.2
Use Euler's equations to simplify $\cos(\theta)\sin(\theta)\cos(2\theta)$.
Solution: By rewriting the trigonometric functions by Euler's equations we get

$$\cos(\theta)\sin(\theta)\cos(2\theta) = \frac{e^{i\theta}+e^{-i\theta}}{2}\frac{e^{i\theta}-e^{-i\theta}}{2i}\frac{e^{2i\theta}+e^{-2i\theta}}{2}$$
$$= \frac{(e^{2i\theta}-e^{-2i\theta})(e^{2i\theta}+e^{-2i\theta})}{8i} = \frac{e^{4i\theta}-e^{-4i\theta}}{8i} = \frac{\sin(4\theta)}{4}.$$

53 Matrices and linear algebra

53.1 Matrices and linear algebra

53.1.1 Handout

Linear systems of equations

As we increase the number of unknown quantities or variables we usually need more equations to decribe the relationships between them. If there are n unknown we normally need at least n equations to determine their values.
When more than one equation is used to describe the relationship between the same variables they are called a *system of equations* or *simultaneous equations* and the operation of using these equations to compute the values of the unkowns is to *solve the system of equations*.
The two main methods used to solve systems of equations are *elimination* and *insertion*. Suppose by way of example that we are given the equations $Ax+By=C$ and $Dx+Ey=F$ where A,B,C,D,E,F are known real numbers.
To solve the system by elimination we multiply one of the equations by a number so as to obtain a common term in both equations. If the number D is not zero we can multiply the equation $Dx+Ey=F$ by the number A/D and obtain the equation $Ax+\frac{EA}{D}=\frac{FA}{D}$. By subtractin the right hand side of this equation from the right hand side of $Ax+By=C$ and similarly for the left side we obtain

$$\begin{aligned} & Ax+By-(Ax+\frac{EA}{D}y) &=& \; C-\frac{FA}{D} \\ \text{iff} \quad & (B-\frac{EA}{D})y &=& \; C-\frac{FA}{D} \\ \text{iff} \quad & y &=& \; \frac{DC-FA}{DB-EA}. \end{aligned}$$

(The notation "iff" is short-hand for "if and only if", which means that the statements on both sides are equivalent.) This can then be put into one of the original equations to find the value of x:

$$\begin{aligned} & Ax+B\frac{DC-FA}{DB-EA} &=& \; C \\ \text{iff} \quad & Ax &=& \; C-\frac{BDC-BFA}{DB-EA} \\ \text{iff} \quad & x &=& \; \frac{BF-CE}{DB-EA} \end{aligned}$$

To solve the same system of equations using insertion we start by isolating either x or y in one of the equations. Using the first this gives $x=\frac{C-By}{A}$. If we insert this x into the second equation we obtain

$$\begin{array}{lrcl} & D\frac{C-By}{A}+Ey &=& F \\ \text{iff} & \frac{DC}{A}-\frac{BD}{A}y+Ey &=& F \\ \text{iff} & (EA-BD)y &=& FA-CD \\ \text{iff} & y &=& \frac{DC-FA}{DB-EA}. \end{array}$$

x is then computed as was done in the earlier case.
The two methods are equivalent and a matter of convenience which one is used at any given time.
Note that it is not particularly useful to try to learn these formulas. It is much more useful to understand the methods used above and apply the method to individual examples.

Example 3.5.1

Solve the system

$$\begin{aligned} 4a+2b-c&=2, \\ a-b+c&=12, \\ a+b+c&=5. \end{aligned}$$

Solution: We start by adding up the last to equations, we get $2a+2c=17$ which implies $c=\frac{17}{2}-a$.
We then add the middle equation two times with the top equation and get $6a+c=26$ and put $\frac{17}{2}-a$ in for c. Then notice:

$$6a+\frac{17}{2}-a=5a+\frac{17}{2}=26 \iff 5a=\frac{35}{2} \iff a=\frac{7}{2}.$$

When a is determined we find $c=\frac{17}{2}-a=\frac{17-7}{2}=5$. By putting that into the last equation we get $c=5-c-a=-a=\frac{-7}{2}$. We have found the solution which is

$$a=\frac{7}{2}, \quad b=\frac{-7}{2}, \quad c=5.$$