Power and sample sizes math612.0 A1: From numbers through algebra to calculus and linear algebra

Gunnar Stefansson (editor) with contributions from very many students

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Power and sample sizes

The power of a test

Suppose we have a method to test a null hypothesis against an alternative hypothesis. The test would be "controlled" at some level α , i.e. $P[reject H_0] < \alpha$ whenever H_0 is true.

On the other hand, when H_0 is false one wants $P[reject H_0]$ to be as high as possible.

If the parameter to be tested is θ and θ_0 is a value within H_0 and θ_a is in H_a then we want $P_{\theta_0}[reject \ H_0] \leq \alpha$ and $P_{\theta_0}[reject \ H_0]$ as large as possible.

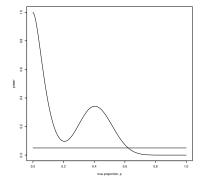
For a general θ we write

$$\beta(\theta) = P_{\theta}[reject \ H_0]$$

for the power of the test

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The power of tests for proportions



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The Power of the one sided z test for the mean

The one sided z-test for the mean (μ) is based on a random sample where $X_1 \dots X_n \sim n(\mu, \sigma^2)$ are independent and σ^2 is known.

The power of the test for an arbitrary μ can be computed as:

$$eta(\mu) = 1 - \Phi\left(rac{\mu_{\mathsf{0}}-\mu}{rac{\sigma}{\sqrt{n}}} + z_{1-lpha}
ight)$$

Power and sample size for the one-sided z-test for a single normal mean

Suppose we want to test $H_0: \mu = \mu_0$ vs $H_a: \mu > \mu_0$. We will reject H_0 if the observed value

$$z = \frac{\overline{x} - \mu_0}{\sigma / \sqrt{n}}$$

is such that $z > z_{1-\alpha}$.

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The non central t - distribution

Recall that if $Z \sim n(0,1)$ and $U \sim \chi^2_{\ v}$ are independent then

$$\frac{Z}{\sqrt{\frac{U}{v}}} \sim t_{v}$$

and it follows for a random sample $X_1 \dots X_n \sim n(\mu, \sigma^2)$ independent; that

$$\frac{\bar{X} - \mu}{\frac{s}{\sqrt{n}}} = \frac{\frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}}}{\sqrt{\frac{\sum (X_i - \bar{X})^2}{\frac{\sigma^2}{n-1}}}} \sim t_{n-1}$$

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The power of t-test for a normal mean (warning: errors)

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Power and sample sizes

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Power and sample size for the one sided t-test for a mean

Suppose we want to calculate the power of a one sided t-test for a single mean (one sample), this can easily be done in R with the power.t.test command.

The power of the 2-sided t-test

A power analysis on a two-sided t-test can be done in R using the *power.t.test* command.

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The power of the 2-sample one and two-sided t-tests

The power of a two sample, one-sided t-test can be computed as follows:

$$\beta_{(\mu_1\mu_2)} = P_{\mu_1\mu_2} \left[\frac{Z + \Delta}{\sqrt{U/(n+m-2)}} > t^*_{1-\alpha,n+m-2} \right]$$

and the power of a two sample, two-sided t-test is give by:

$$\beta_{(\mu_1\mu_2)} = P_{\mu_1\mu_2} \left[\frac{Z + \Delta}{\sqrt{U/(n+m-2)}} > t^*_{1-\alpha,n+m-2} \right] + P_{\mu_1\mu_2} \left[\frac{Z + \Delta}{\sqrt{U/(n+m-2)}} \right]$$

where $\Delta = \frac{(\mu_1 - \mu_2)}{\sigma\sqrt{\frac{1}{n} + \frac{1}{m}}}$ and U is the SSE.

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Sample sizes for two-sample one and two-sided t-tests

The sample size should always satisfy the desired power.

A case study in power

Want to compute power in analysis of covariance:

$$y_{ij} = \mu_i + \beta x_{ij} + \epsilon_{ij}, \ i = 1, 2, \ j = 1, \dots J,$$

where $\epsilon_{ij} \sim n(0, \sigma^2)$ are i.i.d.?

This can be done by simulation and can easily be expanded to other cases.

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