fish5102stockcatch The development of a year-class

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19. desember 2016

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1 The development of a year-class

1.1 Background to the stock and catch equations

We will derive an equation to describe how a specified cohort develops across time.
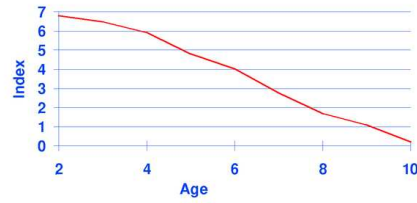
Development of a cod cohort, as seen by the number of cod caught in groundfish surveys in Icelandic waters. Average on log-scale across several years.

1.1.1 Details

If recruitment in a stock is variable and if the species is long-lived, it is necessary to investigate how year-classes in the stock develop. It is not enough to view the stock as an entity or to view the stock by length classes. In fact such information can be highly misleading if no consideration is given to changes in year-class strength and the age composition. In this section the development of a year- class in numbers and weight will be considered.

It should be fairly clear that the age composition of catches and the number of fish by age groups per towing mile gives considerable information about changes in stock size.

The remainder of this section is devoted to deriving an equation which describes how the number of individuals in a yearclass may change across time.

Note 1.1. An equation which follows a cohort across time and describes the changes in population size is referred to as a stock equation.

The most common methods to estimate stock size are based on the assumption that within a specific time period there is constant fishing pressure and constant natural mortality. From its birth, a specific year-class can only decrease in numbers and with constant pressure each fish in the year-class will have a constant probability of death from fishing or natural causes during a short period of time. It is convenient to think of the full time period under consideration as one year, and the time t therefore is in the interval from zero to one. It can then be expected that the proportional decrease of a specific year-class is even throughout the year.

1.1.2 Examples

Example 1.1. The figure above shows the number of fish as an index by age groups in a groundfish survey in the Icelandic economic zone. The cod is used as an example here and it is clear that the number of fish decreases as a function of age. This is natural since it is quite obvious that unless there is considerable immigration into the area the fish must get fewer as they get older. In the graph, a logarithmic scale is used on the y-axis. The remainder of the section clarifies why this is a natural choice.

1.2 Symbols

Notation Numbers at year start and end, respectively : N_0, N_1
Number at specified time, t: N_t
Short time interval: (t_1, t_2)
Length of interval: Δt
Change in stock size during interval: ΔN

1.2.1 Details

Definition 1.1. Notation for stock and catch equations:
N_t = Number of fish in a year-class at time t, where t is $0 \leq t \leq 1$
N_0=Number of fish at the start of the year
N_1=Number of fish at the end of the year
(t_1, t_2)=Time interval such that t_1 is the start of the time interval and t_2 is the end of the time interval
Δt=Change in time between t_1 and t_2
ΔN=Change in stock size between t_1 and t_2

1.2.2 Examples

Example 1.2. The following table lists the averages from indices of different year-classes for cod from 1985-1993:

Age	2	3	4	5	6	7	8	9	10	Ave 5-10
Index	904.0	659.2	370.8	121.9	57.0	16.3	5.4	3.0	1.2	
Log	6.81	6.49	5.92	4.80	4.04	2.79	1.69	1.09	0.21	
Diff.		0.32	0.58	1.11	0.76	1.25	1.10	0.59	0.88	0.95

The conclusion that might be drawn from this is that the total mortality of cod had been about 0.6-1.2 during the period in question. As will be indicated later, there are, however, reasons for checking thoroughly whether the summations of these compilations are valid and whether a sensible average is being estimated. The reason, among other things, is that two strong year-classes appear in some of these numbers but not all of them and have a distorting effect on the overall picture. There is also immigration from Greenland during this period. Therefore, the composition of the year-classes needs to be considered in addition to the age composition.

1.3 Yearclass development during short time interval

Proportion which dies during a time interval =Probability of fish dying

$$\frac{\Delta N}{N} = -Z\Delta t$$

1.3.1 Details

A natural assumption about the change in the number of fish in a cohort is that the numbers should be reduced proportionately, i.e. each fish should have some probability of dying (due to fishing or natural causes) and the overall effect on the yearclass would be that the fraction which dies is fixed.

Further, it is also natural to assume that if the time interval is short enough, the fraction which dies should be proportional to the time interval. Obviously this assumption can not be true if the time interval is long, but for time intervals short enough it should hold.

In summary, therefore, the proportional change in numbers in a short time period, Δt, should be in proportion to the length of the time period.

> **Definition 1.2. Proportional change in population size**:
> $$\frac{\Delta N}{N} = -Z\Delta t,$$

where Z is some number. The change is negative since the yearclass is diminishing in numbers.

1.4 The differential equation

> Differential equation:
> $$\frac{dN}{dt} = -ZN$$

1.4.1 Details

When the previous equation is rewritten and the time interval shortened (formally, we let $\Delta t \to 0$), this leads immediately to a simple differential equation:

$$\frac{dN}{dt} = -ZN$$

Even without any knowledge of differential equations, it should be noted that the logic behind this equation implies that if a short time interval corresponds to halving of the numbers, then twice this time interval will lead to a further 50% reduction and so on. Hence the reduction in numbers will be 1/2 to the power of the number of such time intervals.

Since this is true for any short time interval, it also becomes clear that the solution to this differential equation must be some sort of power function in time, i.e. of the form a^t for some number a.
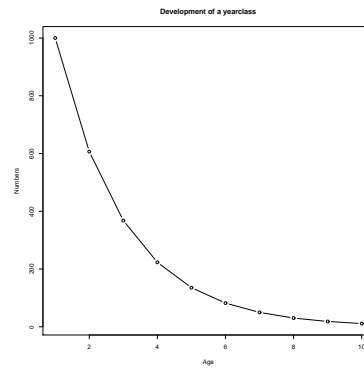
Although the form of the solution is intuitively obvious, formal derivation is required to obtain the correct functional form. This is done using well-known methods of calculus but will not be detailed further here.

1.5 The stock equation

Solution to differential equation:

$$N_t = e^{-Zt} N_0$$

Describes numbers at time t as compared to initial numbers.

1.5.1 Details

The solution to the previous differential equation is the exponential growth function where the size of the year-class at time t (within a year) decreases as an exponential function of time, from the stock size in the beginning (of the year).

Definition 1.3. Single-year stock equation:

$$N_t = e^{-Zt} N_0$$

where Z is the total mortality rate and e^{-Z} is the survival rate

Although it is most common to consider changes within a year, this is not a requirement. In particular it is quite feasible to consider shorter time intervals such as monthly time steps. However, it is not advisable to assume that the stock equation is valid for many years. Thus, the stock equation is most commonly used to connect the yearclass numbers at the beginning of the year (t=0) and the end of the year (t=1).

1.5.2 Examples

Example 1.3. The figure above indicates the development of a yearclass with $Z = 0.5$, from initial numbers of $N = 1000$ in the beginning of the first year.

The form of the curve is typical for stock-reduction curves. Notably, when Z is large, the decline in numbers is quite quick from the initial size to negligible values.

It must be recalled, however, that the decline in numbers is only a part of the story, since the individuals grow and the biomass trend is thus somewhat different from the trend in numbers.

The figure can be generated using R using the following code:

http://tutor-web.net/fish/fish5102stockcatch/lecture10/stock-equation.r

1.6 Development of stock

$$N_{a+1,y+1} = e^{-Z_{ay}} N_{ay}$$

1.6.1 Details

The basic equation is only valid for a single year-class and therefore the equation needs to be modified in such a fashion that the age groups and the years are easily identified.

Naturally the number of fish at the end of the year is the same as the number of fish in the same cohort (year-class) at the beginning of the following year.

Definition 1.4. Stock equation notation:
N_{ay}=Number of fish of age a in the sea at the begining of year y $N_{a+1,y+1}$=Number of fish at the end of the year, i.e. beginning of the following year

Definition 1.5. Across-year stock equation:

$$N_{a+1,y+1} = e^{-Z_{ay}} N_{ay}.$$

1.7 Mortalities work in sequence

Suppose $Z = F + M$...

$$N_1 = e^{-Z} N_0 = e^{-(F+M)} N_0 = e^{-F-M} N_0 = e^{-F} e^{-M} N_0$$

1.7.1 Details

Definition 1.6. Mortality equation:

$$Z = F + M$$

F=Fishing mortality
M=Natural mortality

It should be noted that the different mortalities, as set up here, work naturally in sequence. Thus, the same final yearclass size results whether natural mortality is applied before or after fishing mortality.

This is a consequence of writing the initial mortality process as a differential equation, rather than applying discrete proportional mortalities as is a common method when computing

interest rates.

The mortality resulting from repeated applications of individual mortalities becomes a direct sum of the individual mortalities.

In particular, if the total mortality, Z, can be written as a sum of two mortalities, $Z = F + M$ then

$$N_1 = e^{-Z}N_0 = e^{-(F+M)}N_0 = e^{-F-M}N_0 = e^{-F}e^{-M}N_0$$

and hence it is irrelevant which mortality is applied first - the outcome in terms of the number of survivors is the same.

1.7.2 Examples

Example 1.4. Suppose we start with 1000 fish and kill 50% each year, resulting in the following development of a yearclass:

```
1 1000
2  500
3  250
4  125
5   63
```

Of course this is done by simply multiplying by 0.5 at each stage. Note that $e^{-0.693} = 0.5$ so this corresponds to using a total mortality rate of $Z = 0.693$.

Reduction from 1000 at 1 to 250 at 3 is 75%, i.e. by 0.5*0.5=0.25. It must be noted that the effect is of course not additive (since a doubling of 50% reduction would then be to zero).

But if we use Z=0.693 twice with $e^{-0.693} = 0.5$ then the reduction corresponds to the sum 0.693+0.693=1.386 where the mortality rates simply add up ($e^{-0.693} = 0.5$ and $e^{-1.386} = 0.75$).

1.8 Rates and proportions...

Logarithms or not –

$$1 - e^{-0.2} = 0.18$$

Rates and proportions
Mortality rate $= 0.2$ means 18% die...
Note that Z can be greater than 1!

1.8.1 Details

There is a considerable difference in the interpretation of mortality rates and survival proportion or proportions which die.

> **Definition 1.7. Survival proportion**:
>
> $$e^{-Z}$$
>
> **Proportion of deaths**:
>
> $$1 - e^{-Z}$$

The proportion which survives the year is e^{-Z}. The mortality rate itself can be arbitrarily large (i.e. Z can be greater than 1). However, as Z increases, the fraction which survives, e^{-Z}, goes to zero and the fraction which dies, $1 - e^{-Z}$, goes to one.

1.8.2 Examples

Example 1.5. Since $1 - e^{-0.2} = 0.18$, a total mortality rate of 0.2 means that 18% of the yearclass dies during a year. In particular, assuming the natural mortality rate to be $M = 0.2$ and no fishing takes place, then 18% die due to natural causes per year.

If the mortality rate is 0.8, then $e^{-0.8} = 0.449$, so 44.9% of the yearclass survive the year, i.e. 55.1% die.

Finally, if $Z = 1.2$, then $e^{-Z} = 0.301$ so 69.9% die.

Thus, for low mortality rates, the rate and fraction which die are similar, but as the rate increases, the difference becomes greater (this can also be seen by considering the Taylor series expansion for the exponential function).

1.9 Mortality as log-change

$$\ln N_{a+1,y+1} = \ln N_{ay} - Z_{ay}$$

1.9.1 Details

It should be noted, that on a logarithmic scale there is a convenient relationship between the mortality rate and the changes in stock size:

$$\ln N_{a+1,y+1} = \ln N_{ay} - Z_{ay}.$$

Thus, the mortality rate simply denotes the reduction in the stock size on a logarithmic scale. This will become very useful later, when it will be seen that this equation can be used to estimate mortality rates using simple techniques, if some samples on age composition are available, either from surveys or catches.

1.10   Development of yearclass biomass

Given the stock equation and data on weight and mortality, it is easy to predict the development of biomass.

Notice the maximum biomass at age 8. This would be the harvest if a single cohort was raised in aquaculture.

Development of saithe yearclass
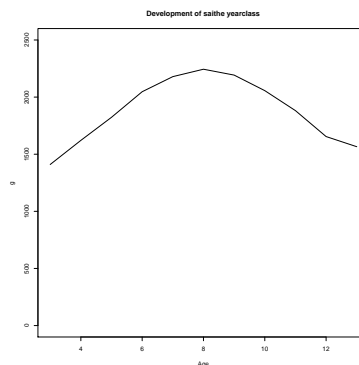
1.10.1   Details

It is of interest to take a specific year-class and investigate how its biomass develops as the year-class grows older. Such an analysis gives an indication of rational utilization of the year-class. Such computations are very simple when the above results are utilized. If, for example, the year-class contains 1000 individuals at the outset, then it will decrease in numbers according to the equation:

$$N_{a+1} = e^{-Z_a} N_a.$$

If there are no catches from the year-class, then the mortality coefficient Z_a simply denotes natural mortality and is usually written M_a. Given the weight at age w_a this can be used to obtain biomass by age.

> **Definition 1.8. Yearclass biomass equation**:
> $B = N_0 \times w_a$

1.10.2   Examples

> **Example 1.6.** The table below contains the mean weight of Icelandic saithe (kg) by age group. It is assumed that natural mortality is M=0.2 and that 1000 individuals enter the stock as 3 year old fish.
> http://tutor-web.net/fish/fish5102stockcatch/lecture10/icelandic-saithe.dat
>
> Given this, one can compute how many fish survive to the following year and thus an estimate is obtained of how many fish from this year-class will be alive at each moment in time. Those counts can then be multiplied by the mean weight at age to obtain the biomass at age. The following figure shows how that biomass changes in total as the individuals grow older.
>
> In this way it is simple to test several different assumptions on natural mortality and mean weights at age in order to investigate whether conclusions on utilization change to a great extent as the assumptions are modified.
>
> The R code to generate the figure is given below:
> http://tutor-web.net/fish/fish5102stockcatch/lecture10/saithe-stock-equation.r

2 Catch curve analysis.

2.1 Introduction

2.1.1 Details

This chapter will deal with the estimation of the mortality of fish in the sea, based on information on the age-composition of the catches.

Obviously using age-composition to estimate mortality is only a very rough approximation. This is especially true for species such as cod which change their behavior after maturity. Similarly, cod have experienced considerable changes in their fishing effort through out the North Atlantic. As a result of their behavioral changes and the alteration to their fishing effort the year-class strength is highly variable and thus, estimating mortality from this data is not always reliable. However, it is still worth while verifying what this method can indicate concerning the mortality of the stock.

2.2 I-Cod catch in nos at age

	1983	1984	1985	1986	1987
3	3.6	6.8	6.5	20.6	11.0
4	10.9	31.6	24.6	20.3	62.1
5	24.3	19.4	35.4	26.6	27.2
6	18.9	15.3	18.3	30.8	15.1
7	17.4	8.1	8.7	11.4	15.7
8	8.4	7.3	4.2	4.4	4.2

A diagonal line in the above table represents a yearclass.

2.2.1 Details

If death rates are stable for a few years it is possible to estimate total mortality almost without any assumptions on natural mortality, since it is usually possible to assume that mortality does not change after a certain age. These assumptions together mean that fishing and natural mortality are constant in time for certain (older) age groups. Care must be exercised before these methods are used, as will be illustrated later.

For simplicity we only consider those age groups where the mortality can be taken to be constant and thus we can simplify the equations, fishing mortality and total mortality respectively, by writing $F = F_{ay}$ and $Z = Z_{ay}$.

Definition 2.1. Basic catch-curve equation:

$$C_{ay} = \frac{F}{Z}(1 - e^{-Z})N_{ay}$$

Given a basic catch by age and year equation, i.e. basic catch-curve equation, the catch for a particular year-class in the next year $(y + 1)$ is given by:

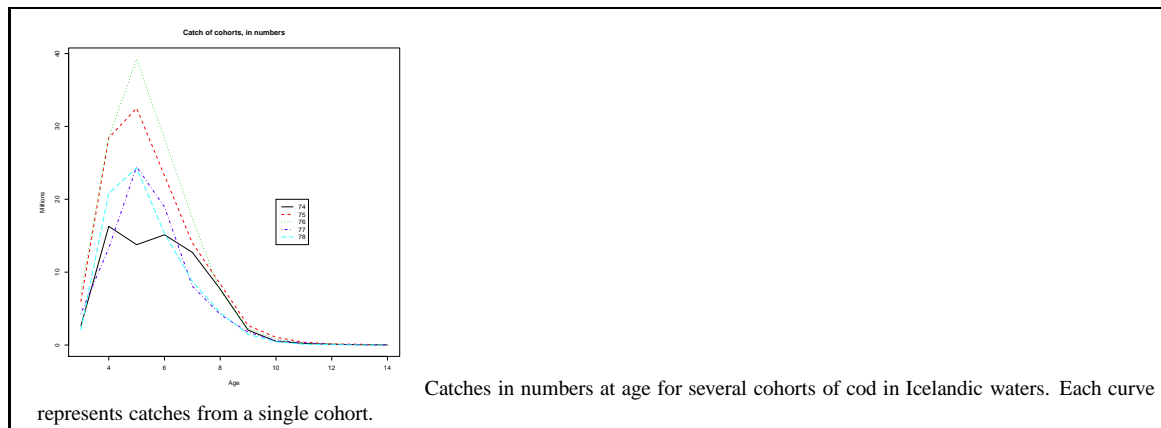$$C_{a+1,y+1} = \frac{F}{Z}(1 - e^{-Z})N_{a+1,y+1}$$

i.e.
$$C_{a+1,y+1} = \frac{F}{Z}(1 - e^{-Z})e^{-Z}N_{a,y}$$

Now, if the equations for the years y and $y+1$ are compared, it is seen that the only difference in the two right hand sides is e^{-Z}.

> **Definition 2.2. Catch-curve equation for $y+1$:**
>
> $$C_{a+1,y+1} = e^{-Z}C_{a,y}$$

2.3   Catch curve analysis (A) Start with some catch curves

Catches in numbers at age for several cohorts of cod in Icelandic waters. Each curve represents catches from a single cohort.

2.3.1   Details

For constant death rates, the catch equation for the following year for a specific year-class can be written in the log-scale.

> **Definition 2.3. Log-transformed Catch-curve equation for $y+1$:**
>
> $$\ln C_{a+1,y+1} = \ln C_{a,y} - Z$$

This means that the fixed death rate can be estimated using the fixed death rate equation.

> **Definition 2.4. Fixed death rate equation:**
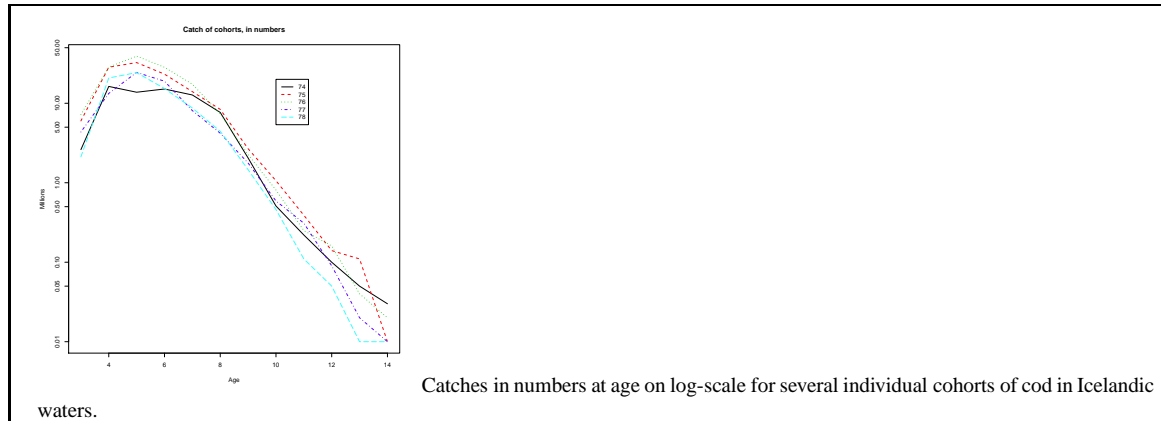>
> $$Z = \ln C_{a,y} - \ln C_{a+1,y+1}$$

2.3.2   Examples

2.4   Catch curve analysis (I-cod) (B) Log-scale (yearclasses 74-78)

Catches in numbers at age on log-scale for several individual cohorts of cod in Icelandic waters.

2.4.1   Details

It has been stated previously that more information can be obtained from catches at age by log-transforming the data. Using the logarithmic scale when the effort is fixed allows the numbers to reduce in a linear fashion.

2.4.2   Examples

Example 2.2. Taking again the yearclasses of cod in Icelandic waters, the log-transformed number become:
http://tutor-web.net/fish/fish5102stockcatch/lecture20/ln-catch-at-age-cod.dat

Example R code:

```
matplot(ages,mat,type='l',lty=1:5,col=1:5,lwd=2,xlab="Age",
ylab="Millions",main="Catch␣of␣cohorts,␣in␣numbers",log="y")
legend(10,20,74:78,lty=1:5,col=1:5,lwd=2)
```

2.5 Catch curve analysis (C) Z-by age : ln(N1)-ln(N0)

Differences on log-scale of catches in numbers at age for several cohorts of cod in Icelandic waters. Note the erratic behaviour of the differences.
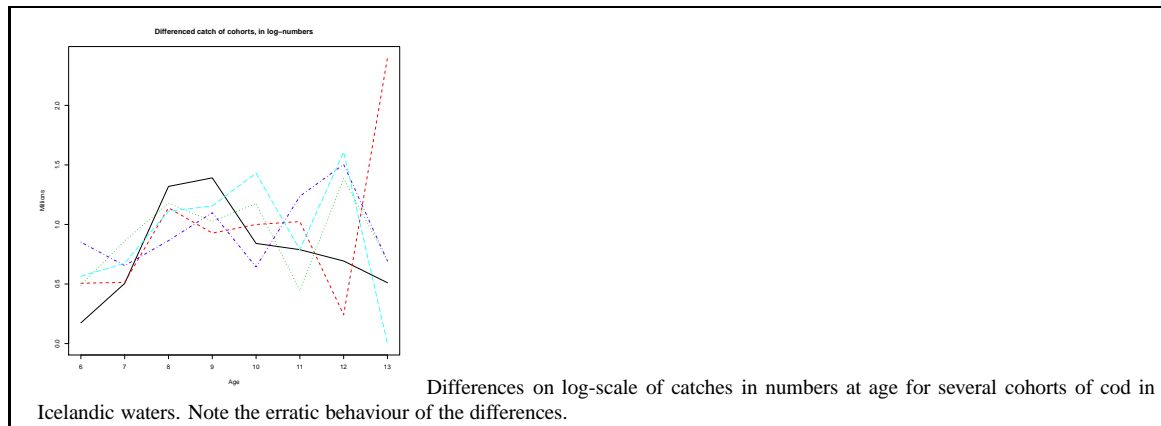
2.5.1 Details

The difference in the log-catch from one year to the next gives an indication of the total mortality. However, the noise in the data makes it difficult to obtain annual Z values, as tempting as it may be.

2.5.2 Examples

Example 2.3. The following table gives such an overview for cod in Icelandic waters:

http://tutor-web.net/fish/fish5102stockcatch/lecture20/z-catch-at-age-cod.dat

It therefore appears that the total mortality was 0.8-1.1 for the oldest fish when these year-classes were 9-14 years old, i.e. during the period from 1984-1992. It is clear that the youngest fish are not completely included in the fisheries because total mortality can not be lower than zero.

It is simplest to investigate the coherence of this analysis in a few simple plots. First, the catch curves are plotted to investigate where they resemble straight lines on a logarithmic scale. Catch curves of this type are common and they are usually used to investigate deviations from assumptions or simply whether the logarithm of a catch in numbers drops roughly linearly.
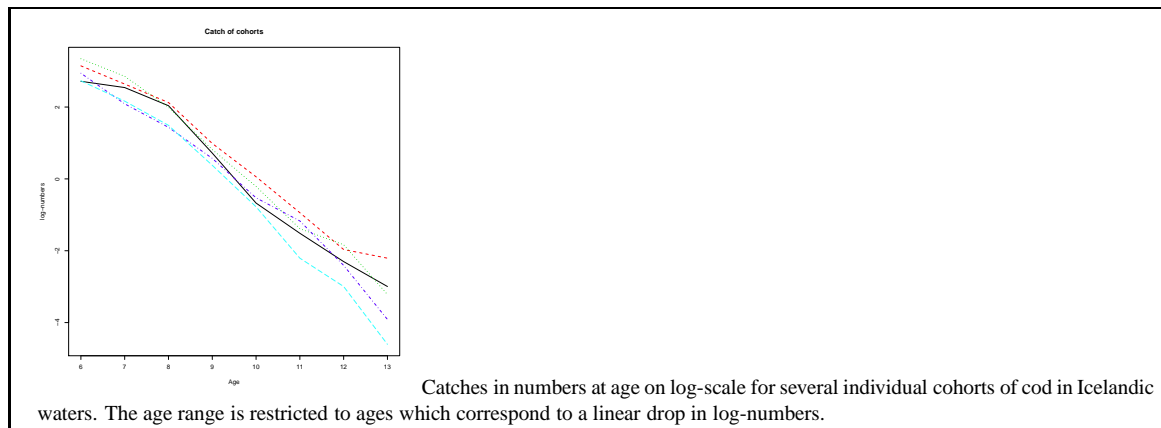
The curves seem to be fairly close to straight lines after about age eight. Therefore the Z is computed and plotted by age groups after the age of eight.

These curves appear at first sight to be reasonable and to vary about a common mean. It is clear, however, that there is considerable variation in the estimate of Z for fish at age 11 or more, which makes sense since there are few fish of that age caught. An estimate of Z could therefore be based on 8-10 year old fish, which gives Z an average of about 1.0. These analyses have been based on year-classes 1974-1978, and therefore this Z applies to the oldest fish during the time period 1982-1988. Therefore the fishing mortality may have been about 0.8 for the oldest fish during the years 1982-1988. As has already been indicated, an increase in fishing effort will lead to an underestimate of total mortality.

Example R code:
http://tutor-web.net/fish/fish5102stockcatch/lecture20/catch-at-age-cod.r/

2.6 Catch curve analysis log axis...

Catches in numbers at age on log-scale for several individual cohorts of cod in Icelandic waters. The age range is restricted to ages which correspond to a linear drop in log-numbers.

2.6.1 Details

Conducting a catch curve analysis on the log axis is usually done by going through the following steps:

1. A specific year-class is selected.

2. The number of landed fish from this year-class , C_{ay}, is plotted against age.

3. The corresponding plot is done on a logarithmic scale, i.e. $\ln C_{ay}$ is plotted against age.

4. Z is computed for each age group and plotted against age.

5. Visual inspection is used to verify at which age Z has become stable and only older fish is considered in what follows.

6. A regression is performed for older fish of $\ln C_{ay}$ against a, The slope from this regression gives an estimate of Z.

7. It is checked, whether deviations exist from a linear relationship. Such deviations would be an indication of, for example, a change in the selection pattern. This can be investigated, for example, by plotting the deviations from the line against age.

WARNING: (1) If fish change their behavior at maturation it is very likely that this method will not work until after the fish has reached 100% maturity. (2) If effort steadily increases over the time period this method will result in pure and utter nonsense. In fact, an increase in effort would result in a bigger catch in the following year than would be obtained with a stable effort. This leads to the result that an estimate of Z will be lower (the decrease in catches is not as great) and thus an increasing effort leads to an underestimate of the mortality rate with this method. (3) The method is sometimes used on an average (in logarithms) of some years or year-classes. In this case, special care must be taken not to include a differential number of ages from year-classes which are very strong or very weak, for example if a strong year-class is only included for the first two youngest ages then this may clearly give misleading results.
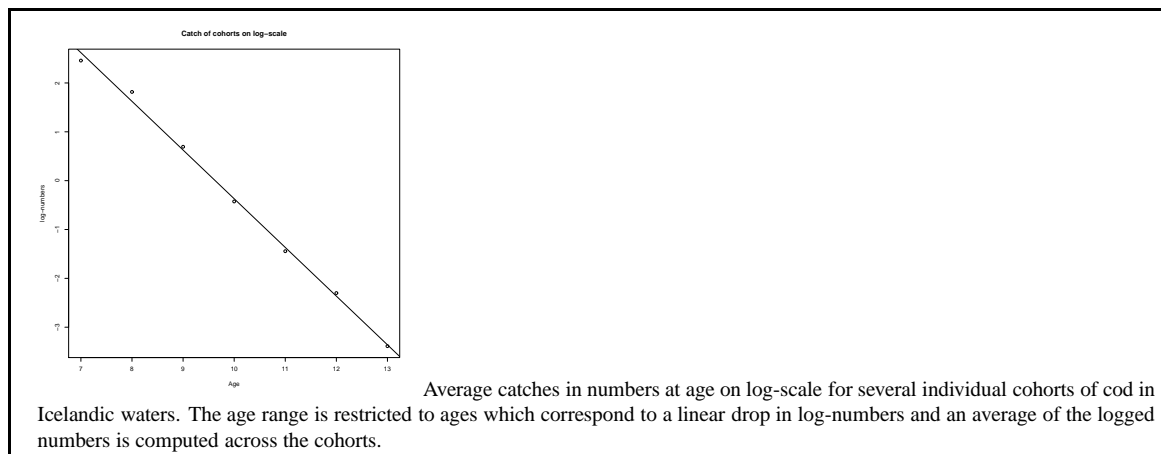
2.6.2 Examples

2.7 Catch curve analysis (D) Z estimated as 1.04

Average catches in numbers at age on log-scale for several individual cohorts of cod in Icelandic waters. The age range is restricted to ages which correspond to a linear drop in log-numbers and an average of the logged numbers is computed across the cohorts.

2.7.1 Details

If there is an age range where the log-catches appear to be linear, when viewed as a function of age, then this can be used as data in a linear regression. Notably, if log-catch is regressed on age, the negative of the resulting slope is an estimate of total mortality, Z, i.e the slope=-Z.

2.7.2 Examples

average Z value for each year.

Example R code:
http://tutor-web.net/fish/fish5102stockcatch/lecture20/ave-z-cod.r

2.8 Catch curve analysis

To compute average of log-catch by year-class the following must be met:

- Mortality must not increase by age

- Effort must not increase by time

- Averaging needs to be carefully implemented

- Catch needs to be proportional to stock size

2.8.1 Examples

Example 2.6. In the cod example, when more detailed analysis is applied it appears that the total mortality reduced to some extent during the years after 1983, but then a considerable increase appears to have a occurred in 1988. Therefore, it seems clear that there is a potential for misleading results by computing a single mortality rate for the whole time period.

2.9 I-cod, Yearclasses 1960-92.
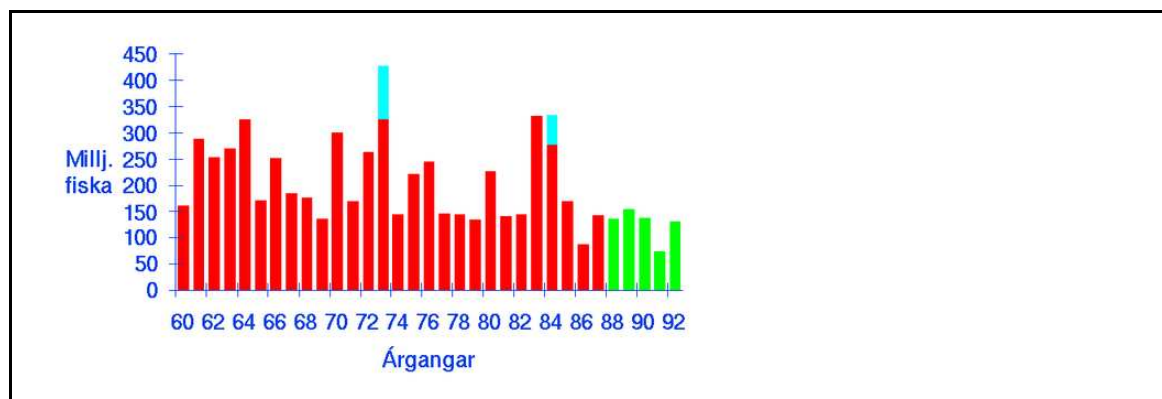
2.9.1 Details

Although it is interesting to be able to estimate total mortality, this is not going to be enough since it does not give information about stock size or structure. In particular there is no information about year-class size, which in most stocks is a primary driving force in the population dynamics.

Note that the catch-curves alone does NOT give information about absolute biomass or absolute numbers, but we need recruitment to estimate yield etc.

3 Natural mortality

3.1 The natural mortality rate

Fishing mortality and natural mortality need to be untangled.

The equation for total mortality can be written such that fishing and natural mortality are separate but linked:

$$Z_{ay} = F_{ay} + M_{ay}$$

Sources of natural mortality include predation, disease etc.

3.1.1 Details

There are considerable difficulties in estimating natural mortality. In the worst cases this can be based completely on guesswork and at best it is possible to obtain an estimate with a high variance. For this reason it is necessary to test the effect of this coefficient in terms of advice given on the utilization of the stock and in terms of the stock estimate. Thus, the effect of a considerably incorrect M on advice is often tested, but within a reasonable framework. In such cases, the tests are based on reasonable limits which include the restriction that M is not inconsistent with available data.

In some cases it is possible to look at time periods under little or no fishing. Here one can use the total mortality as an estimate of natural mortality.

Note 3.1. In general, we would define natural and fishing mortalities as a decomposition of the total mortality which we can estimate from the slope of catch curves.

3.1.2 Examples

Example 3.1. If natural mortality is 0.1 and there is no fishing from the stock, then the stock is reduced through the year to the fraction

$$e^{-0.1} = 0.90$$

of the original size, or to 90% of the size at the beginning of the year. Thus a specific year-class gets reduced by about 10% per year and in 10 years it gets reduced to

$$e^{-10*0.1} = e^{-1} = 0.37$$

or to 37% of the original size. Thus, there is still a considerable proportion left of a big year-class at the age of 10 years if M is 0.1. If, on the other hand, M is 0.2, then there are 14% left in 10 years and if M is 0.3, there are only 5% alive in 10 years time. These considerations put some bounds on what values of M are "reasonable" for a specific stock.

It should be noted that although usually a fixed natural mortality coefficient is used, this does not mean that anyone considers such an assumption to be a reasonable one nor is

this an essential prerequisite for the computation to give the correct picture. For example, it is well known that if natural mortality is variable about some mean then a fairly good approximation is obtained by using this mean in place of the individual annual values (Ulltang, 1976). Thus, a summation of the form "M is 0.2"is roughly equivalent to saying "M is on average close to 0.2".

3.2   On natural mortality, M
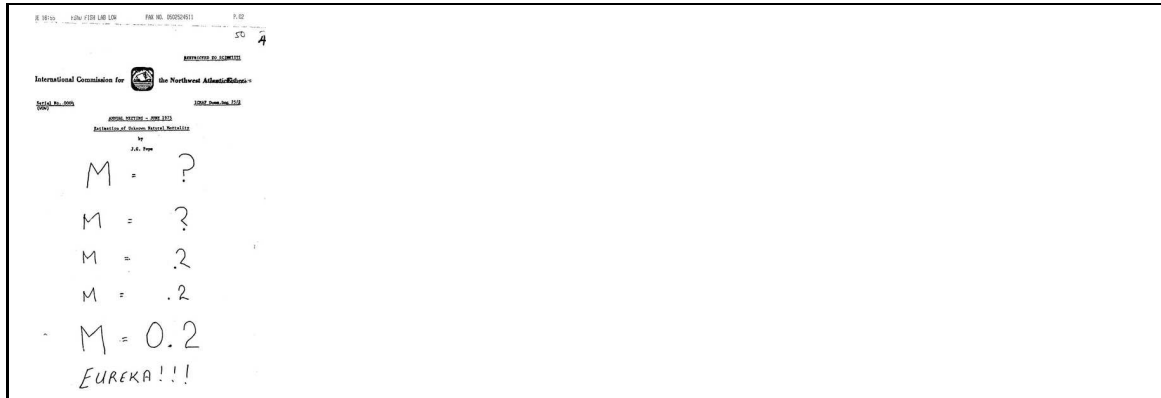
3.2.1   Details

Natural mortality is of course an important component of many assessments and the population dynamics of a stock may depend heavily on natural mortality.

The figure above gives one approach to estimating natural mortality – simply rewrite 0.? often enough that it changes to 0.2.Although it may be claimed that this approach has been used in some situations, it is usually possible to do a little better.

3.3   More on M
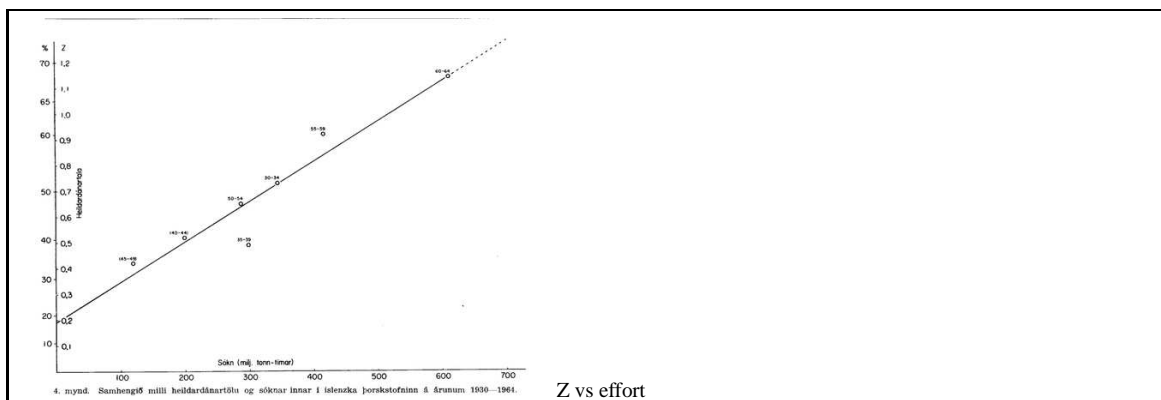
Z vs effort

3.3.1   Details

The above plot shows how natural mortality has been estimated by plotting total mortality (obtained from a catch curve analysis) on the y-axis and an effort measure on the x-axis. The figures effort measure could be the number of boat-days or ton-days etc. The intercept of the line with the y-axis provides an estimate of natural mortality.

When applicable, this type of figure provides the potential to separate the total mortality into that due to fishing and that due to natural causes. It must be noted, however, that the natural mortality estimated in this fashion is only a long-term average across the years (and ages) in question. Whether this is useful or not will depend on the species under consideration.

It should also be noted that when a species has been heavily exploited for a long time, the effort data on the x-axis will only be available as high values and no data will exist close to the origin on the x-axes. It follows that any estimate of the intercept will be highly uncertain and M is therefore a fairly poor estimate.

3.4 Additivity of mortalities

3.4.1 Details

Definition 3.1. Notation for natural death: D_{ay}

An equation can be derived to describe number of natural deaths by proceeding in the same way as was done for the catch equation. Begin by assuming that the natural mortality in numbers during a short time period is proportional to the stock size and the time interval.

Definition 3.2. Proportionality constant for natural morality notation: M_{ay}

Given the natural death and proportionality constant for natural mortality, an equation can be written to describe the number of deaths from "natural"causes.

Definition 3.3. Natural death equation:

$$D_{ay} = \frac{M_{ay}}{Z_{ay}}(1 - e^{-Z_{ay}})N_{ay}$$

Note that it has not been assumed that this M is related to the earlier definition of natural mortality. From this equation it is clear that the fish that die during the year add up to

$$C_{ay} + D_{ay}$$

.
If the equations which describe these quantities are added together, the following is obtained:

$$C_{ay} + D_{ay} = \frac{F_{ay}}{Z_{ay}}(1 - e^{-Z_{ay}})N_{ay} + \frac{M_{ay}}{Z_{ay}}(1 - e^{-Z_{ay}})N_{ay}$$
$$= \frac{F_{ay} + M_{ay}}{Z_{ay}}(1 - e^{-Z_{ay}})N_{ay}$$
$$= (1 - e^{-Z_{ay}})N_{ay}$$

where the final result is true if and only if $Z = F + M$, i.e. if the M-value here has the same meaning as before. This is in accordance with the survival probabilities and the probabilities of dying which were derived above.

A fish which is alive at the beginning of this year will either be caught, die due to natural causes, or survive the year. Now these equations can be added up in order to check whether the system of the equations is internally consistent:

$$C_{ay} + D_{ay} + N_{a+1,y+1} = (1 - e^{-Z_{ay}})N_{ay} + N_{a+1,y+1}$$
$$= (1 - e^{-Z_{ay}})N_{ay} + e^{-Z_{ay}}N_{ay} = N_{ay}$$

The various equations thus seem to match: The right hand side indicates the number of fish available at the beginning of the year but the left hand side is a summary of the possible fate of the individual fish.

3.5 Effects of different mortality rates

3.5.1 Examples

Example 3.2. The effect of the different mortality types is illustrated in the following table.

Total mortality	0.20	0.40	0.60	0.80	1.00	1.20
Natural mortality	0.20	0.20	0.20	0.20	0.20	0.20
Fishing mortality	0.00	0.20	0.40	0.60	0.80	1.00
Proportion surviving	0.82	0.67	0.55	0.45	0.37	0.30
Proportion killed	0.18	0.33	0.45	0.55	0.63	0.70
Natural deaths	0.18	0.16	0.15	0.14	0.13	0.12
Proportion caught	0.00	0.16	0.30	0.41	0.51	0.58
Natural deaths	0.18	0.16	0.15	0.14	0.13	0.12

4 Stock and catch equations

4.1 Deriving the catch equation

Need to derive an equation describing the catch in numbers in relation to stock size.

4.1.1 Details

As before, we start by considering only a single cohort and developing a model to describe how this year-class is fished. The figure shows how fishing starts from a typical year-class of Icelandic cod at the age of 3 and the catch in numbers increases from the cohort until a maximum is attained. After this there is a steady decrease in numbers from this year-class in the catches.

4.2 Development of catch from a cohort

Constant fishing intensity within a year or during short time interval needs to be assumed to determine the amount of catch from a cohort.

Catch in proportion to stock and length of timestep:

$$\Delta C = FN\Delta t$$

4.2.1 Details

Assume that the fishing intensity is constant within the year. Then the catch from the cohort during a short period should be proportional to the stock size and the length of the time period.

Definition 4.1. Proportional change in catch equation:

$$\Delta C = FN\Delta t,$$

F = some number

4.3 Development of catch from a cohort

As before this leads to a simple differential equation:

$$\frac{dC}{dt} = FN = Fe^{-Zt}N_0$$

where the previous results concerning the development of a stock within a year have been used.

4.3.1 Details

As before this leads to a simple differential equation:

$$\frac{dC}{dt} = FN.$$

Recall from the previous results concerning the development of a stock within a year that the stock equation links the numbers, N (or N_t), at time t to the numbers at the beginning of the year, N_0 via

$$N = e^{-Zt}N_0.$$

This equation can be substituted into the equation describing the catches to obtain a description of the change in catches per unit time.

Definition 4.2. Change in catches per unit time equation:

$$\frac{dC}{dt} = Fe^{-Zt}N_0.$$

Although the earlier descriptions of stock development led to a differential equation which needed to be solved, the catch description is much simpler since the stock equation has already been solved. Thus, this is now an equation which has a fully defined function on the right-hand side, and this function can simply be integrated to obtain the catch.

4.4 Deriving the catch equation

Integrating gives the annual catch in numbers from that cohort

$$C = \int_0^1 Fe^{-Zt}N_0 dt = \frac{F}{Z}(1 - e^{-Z})N_0$$

4.4.1 Details

By integrating the equation for the derivative of the catches, a formula is obtained for the total catch in numbers from that cohort in a single year.

Definition 4.3. Total cohort catch in a year equation:

$$C = \int_0^1 Fe^{-Zt}N_0 dt = \frac{F}{Z}(1 - e^{-Z})N_0$$

This result will form a cornerstone for all methods which utilize information on the catches in numbers obtained from individual cohorts.

4.5 The catch equation

The catch equation:

$$C = \frac{F}{Z}(1 - e^{-Z})N_0$$

$$C = \frac{F}{Z}\left(1 - \boxed{e^{-Z}}\right)N_0$$
Surviving proportion

$$C = \frac{F}{Z}\boxed{1 - e^{-Z}}N_0$$
Proportion that dies

$$C = \boxed{\frac{F}{Z}(1 - e^{-Z})}N_0$$
Proportion which gets caught

4.5.1 Details

The catch equation is an extremely important concept and is used in most modern assessment methods.

It should be noted that the catch equation $C = \frac{F}{Z}(1 - e^{-Z})N_0$ can be read in terms of the survival fraction, fraction which dies and fraction harvested of fraction which dies.

Recall that $e^{-Z}N_0$ is the number which survives the year and therefore e^{-Z} is the fraction which survives. It follows that the fraction which dies during the year is $(1 - e^{-Z})$. Finally, the ratio $\frac{F}{Z}$ is that fraction of those who die which gets caught.

The form of this equation does not indicate the possibility of several age groups and years. Rather, the equation describes how a single age group gets caught during a single year. Thus, it is customary to omit any subscripts since they are not needed.

4.6 Change in yearclass size during a year

$$N_1 = e^{-Z}N_0$$

4.6.1 Details

Definition 4.4. Equation for annual change in population size:

$$N_1 = e^{-Z}N_0$$

F=fishing mortality rate
Z=total mortality rate

Remember that the catch is a fraction (F/Z) of what dies.

The fishing mortality rate is a measure of what proportion of the stock is killed due to fishing, but the mortality rate itself is not a proportion and can be greater than 1. These coefficients are variable from year to year due to changes in effort.

The coefficients are also variable by age. Firstly, the fishing gear does not catch all age groups equally. For example, the smallest fish will escape through the mesh and further, shoaling behavior during spawning may have the effect that the largest fish can be either more or less catchable than the average size fish.

It is therefore necessary to work with an entire table of fishing mortality rates, F_{ay}.

4.7 The catch equation for many age groups and years

Catch equation for many yearclasses

$$C_{ay} = \frac{F_{ay}}{Z_{ay}}(1 - e^{-Z_{ay}})N_{ay}$$

4.7.1 Details

During any given year there will be many cohorts in the catches. Several years will also be considered at the same time and therefore the notation must be extended to indicate the different years and ages.

Definition 4.5. Catch equation for many year classes:

$$C_{ay} = \frac{F_{ay}}{Z_{ay}}(1 - e^{-Z_{ay}})N_{ay}$$

F_{ay} = fishing mortality (rate)

The natural mortality (rate) is defined by $M_{ay} = Z_{ay} - F_{ay}$, and hence the total mortality is the sum of the other two: $Z_{ay} = F_{ay} + M_{ay}$.

The following is a discussion of these mortality coefficients and their relationship in explaining survival probabilities and overall mortality.

Natural mortality coefficients

When estimating a stock size it is usually assumed that natural mortality (M_{ay}) is known. Thus, it is assumed that information on the maximum age or tagging results, acoustic surveys or predation is used to estimate this coefficient. If very young fish are taken into the stock estimate or if considerable spawning mortality exists it is necessary to have M_{ay} vary by age. For stock such as capelin it is desirable to have the coefficients also vary in time.

When the catch is mostly based on larger fish, or where there is little or no variation in predation from year to year and mortality due to spawning is not considerable, it is usually assumed that one natural mortality coefficient is valid for all age groups and years. The labeling of years and ages is usually omitted and the natural mortality coefficients simply denoted by M.