Linear programming

math121-1-linprog Introduction to linear programming

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Linear programming

Optimization projects are quite common, i.e. one needs to find a best numerical solution.

This is usually either a request for finding a maximum or a minimum and sometimes the function to be optimized is linear.

Sometimes one has bounds on the solutions and these are sometimes linear.

Example: Suppose we want to make animal feed as cheaply as possible. We have two feed mixed which we can use. Mix A, costs 10 kr/kg, but mix B costs 20 kr/kg.

The two mixes are different so the energy content of mix A has 500 Kcal but B has 4000 Kcal/kg. Mix A has a lot of protein, 200g per kg of mix, B has 100g per kg.

A farmer wants to combine the two feeds and minimize costs, but still fulfill the needs of the animals. They need at least 2000 Kcal/day and 400g of protein per day.

What is the optimal mix of the two feeds?

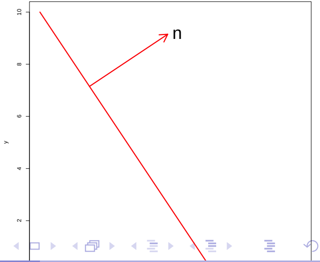
Linear objective function

Want to maximize a function of e.g. 2 variables:

$$z = ax + by$$

but there may be many more variables.
The objective function is a constant, $z = z_0$, if $ax + by = z_0 \Rightarrow$ on a straight line.
The objective function increases in the direction of the normal vector, $\mathbf{n} = (a, b)'$.

Example: Suppose we want to produce two products, A and B where the profit is 3kr of each unit of product A but 2 kr per unit of product B. If we produce a total of x units of A and y units of B then the total profits are 3x+2y.



The solution set

The set may be empty

The set may be unbounded

There may be a unique solution to the problem

The general form

In matrix and vector notation:

$$min_x c' x$$

w.r.t.

$$Ax \leq b$$

Shadow values

At the optimum, the value of the objective functions (z^*) of a linear programming problem is attained with some constraints aactive, i.e. $\mathbf{a}_i'\mathbf{x}^* = b_i$, but others may not be, i.e. $\mathbf{a}_i'\mathbf{x}^* < b_i$.
One can view the optimum value of the objective function as a function of these constraints $z^* = f(b_1, b_2, \ldots)$.
When we relax the constraints, the solution can only improve.
The **shadow value** of a constraint is the answer to "how much does z^* increase if we relax $\mathbf{a}_i'\mathbf{x}^* \leq b_i$ by one unit".

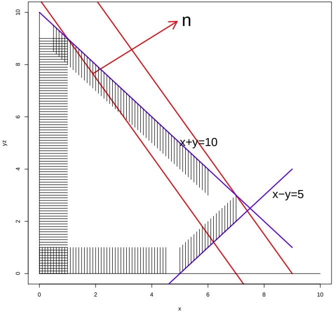
We can write this as $\frac{\partial z^*}{\partial b_i}$.

A visual solution

$$min_x c'x$$

w.r.t.

$$Ax \leq b$$



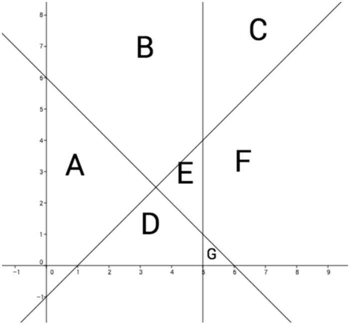
GeoGebra is an excellent tool to draw the figures https://www.youtube.com/watch?v=qjDG940t-PA&feature=relmfu

Example (from a 2004 exam in Iceland) Draw a figure and use it to find the values, x and y, which minimize $z = 2x - y$ with respect to

$$\begin{aligned} 4x + 3y &\leq 12 \\ -x + y &\leq 2 \\ x + 2y &\geq 2 \\ x &\geq 0 \end{aligned}$$

An example

$$
\begin{array}{lrcrcl}
\max z = & 3x & - & 4y & & \\
w.r.t. & x & - & y & \leq & 1 \\
 & x & & & \leq & 5 \\
 & x & + & y & \leq & 6 \\
 & & & x & \geq & 0 \\
 & & & y & \geq & 0
\end{array}
$$



Orðalisti

* Línuleg bestun

An example