On functions of two variables

(MATH221.0: 0 Functions of two variables)

Gunnar Stefansson

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Extensions of univariate functions

One can extend the univariate case of $f : \mathbb{R} \to \mathbb{R}$ in several ways.
Here we will consider only the simplest case: $f : \mathbb{R}^2 \to \mathbb{R}$ so $f(x, y) \in \mathbb{R}$.

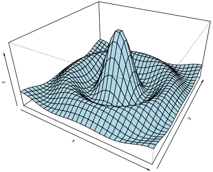

Figure: The function $f(x, y) = sin(x^2 + y^2)/(x^2 + y^2)$, plotted as a surface of points $(x, y, z) \in \mathbb{R}$ with $z = f(x, y)$.

Examples:
$f(x, y) = x + y$; $f(x, y) = x^2 + y^2$
$f(x, y) = xy^2$ – check how this behaves as a function of y for different x-values.
$f(x, y) = (x - y)^2$; $f(x, y) = x/y$ if $y \neq 0$
A popular functions is:

$$f(x, y) = sin(x^2 + y^2)/(x^2 + y^2)$$

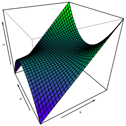

Figure: The surface $z = xy^2$.

Investigating one variable at a time

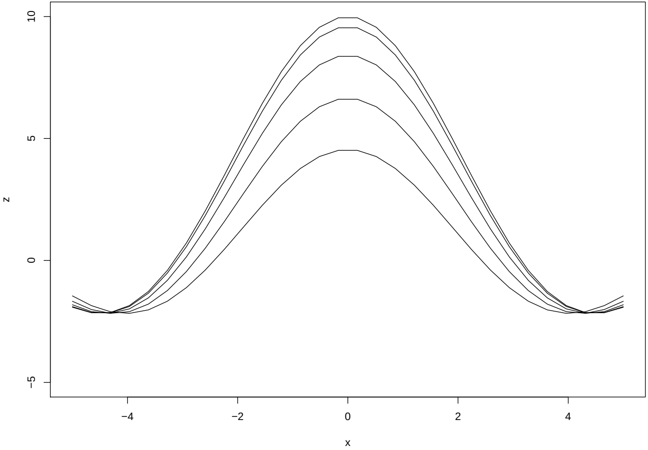

Figure: Plots of a function $z = f(x, y)$, viewed as a function of x, for several fixed levels of y.

Contour plots

A contour plot is a set of points of the form

$$\{(x, y) \in \mathbb{R}^2 : f(x, y) = c\}$$

for some number c.

Example: Plot the contours of $2x + 3y$, $x^2 + y^2$, ...

The equation $F(x, y) = c$

$F : \mathbb{R}^2 \to \mathbb{R}$
Then $F(x, y) = c$ defines a relationship between x and y.
Note that this is a contour of the function.
We can sometimes solve this equation to write y as a function of x.
We can also differentiate this equation...

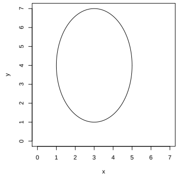

Figure: Points $(x, y) \in \mathbb{R}^2$ satisfying $\frac{(x-3)^2}{4} + \frac{(y-4)^2}{9} = 1$.

Example: $F(x, y) = x^2 + y^2$ etc
See also example with this slide.
If $xy = \arctan(y)$ then we can write $y = f(x)$ and then compute $\frac{dy}{dx}$ by differentiation both sides of the equation and solving for $f'(x)$.
See also `https://www.math.ucdavis.edu/~kouba/CalcOneDIRECTORY/implicitdiffdirectory}`

Partial derivatives

A function of two or more variables can be inspected as a function of one variable at a time:
Suppose $F : \mathbb{R}^2 \rightarrow \mathbb{R}$ is differentiable in each variable.
We write $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$ for the two derivatives.
A local maximum (or minimum) must have

$$\frac{\partial F}{\partial x} = 0$$

and

$$\frac{\partial F}{\partial y} = 0$$

but this may still not be a maximum or a minimum.

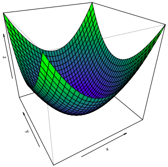

Figure: The function $F(x, y) = x^2 + y^2$.