More on real-valued functions of two variables

math221.1 0 Applied calculus of two variables

Gunnar Stefansson

November 12, 2015

Real functions of more than one variable

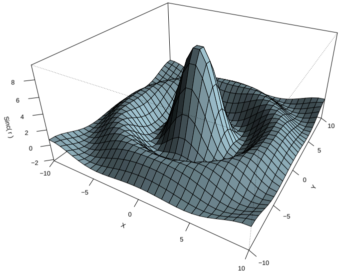

Figure : The function $\sin(x^2 + y^2)/(x^2 + y^2)$.

Typical:

$$f : \mathbb{R}^2 \to \mathbb{R} \; f(x, y) = x^2 + y^2$$

$$g : \mathbb{R}^3 \to \mathbb{R} \; g(x, y, z) = xyz$$

Partial differentiation

In principle, just differentiate with respect to one variable at a time. Write

$$\frac{\partial f(x,y)}{\partial x}$$

$$\frac{\partial f(x,y)}{\partial y}$$

To be differentiable, these partial derivatives need to satisfy criteria...if the partial derivatives are continuous, then the function is differentiable.

The gradient

If $f : \mathbb{R}^n \to \mathbb{R}$, then we define the **gradient** of f as the vector

$$\nabla f(\mathbf{x}) = \begin{bmatrix} \frac{\partial f(x_1, x_2, \ldots, x_n)}{\partial x_1} \\ \frac{\partial f(x_1, x_2, \ldots, x_n)}{\partial x_2} \\ \vdots \\ \frac{\partial f(x_1, x_2, \ldots, x_n)}{\partial x_n} \end{bmatrix}$$

Example: Consider the function $f(x, y) = x^4 + x^2(1 - 2y) + y^2 - 4x + 4$. The gradient of this function at a general point (x, y) is

$$\nabla f(\mathbf{x}) = \begin{bmatrix} \frac{\partial f(x,y)}{\partial x_1} \\ \frac{\partial f(x,y)}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 4x^3 + 2x(1 - 2y) - 4 \\ 2y - 2x^2 \end{bmatrix}$$

Hence e.g. at $(x, y) = (0, 1)$ we can calculate the gradient at this particular point as

$$\nabla f(\mathbf{x}) = \begin{bmatrix} -4 \\ 2 \end{bmatrix}$$

and we can identify any potential maxima or minima as the points where $\nabla f = \mathbf{0}$, i.e. where both $0 = \frac{\partial f}{\partial x} = 4x^3 + 2x(1 - 2y) - 4$ and $0 = \frac{\partial f}{\partial x} = 2y - 2x^2$. For this to occur we need

Higher order derivatives

If the functions are differentiable in the coordinates then we can keep on differentiating to get mixed derivatives...

Example: For a function of only two variables we can compute

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$

and

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

Example: Consider the function ...

The Hessian matrix

The Hessian matrix is the matrix of all combinations of second-order derivatives, for example:

$$H = \begin{bmatrix} \frac{\partial^2 f(x,y)}{\partial x^2} & \frac{\partial^2 f(x,y)}{\partial y \partial x} \\ \frac{\partial^2 f(x,y)}{\partial x \partial y} & \frac{\partial^2 f(x,y)}{\partial y^2} \end{bmatrix}$$

Example: Consider the function $f(x,y) = x^4 + x^2(1-2y) + y^2 - 4x + 4$. The gradient of this function at a general point (x,y) is

$$\nabla f(\mathbf{x}) = \begin{bmatrix} \frac{\partial f(x,y)}{\partial x_1} \\ \frac{\partial f(x,y)}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 4x^3 + 2x(1-2y) - 4 \\ 2y - 2x^2 \end{bmatrix}$$

Hence e.g. at $(x,y) = (0,1)$ we can calculate the gradient at this particular point as

$$\nabla f(\mathbf{x}) = \begin{bmatrix} -4 \\ 2 \end{bmatrix}$$

and the Hessian is

$$H = \begin{bmatrix} \frac{\partial^2 f(x,y)}{\partial x^2} & \frac{\partial^2 f(x,y)}{\partial y \partial x} \end{bmatrix} = \begin{bmatrix} 12x^2 + 2(1-2y) & -4x \end{bmatrix}$$