# Maxima and minima of real-valued functions of two variables

## math221.1 0 Applied calculus of two variables

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# Unconstrained local optimization

Local extrema must satisfy

$$\nabla f(x, y) = 0$$

(if the derivatives exist everywhere)

# Classification of extrema

If $\nabla f(x_0, y_0) = 0$, $H$ the Hessian with eigenvalues $\lambda_1 > \lambda_2$.

- $\lambda_1 > \lambda_2 > 0$: local minimum
  $\Leftarrow det(H) > 0,\ tr(H) > 0$
- $0 > \lambda_1 > \lambda_2$: local maximum
  $\Leftarrow det(H) > 0,\ tr(H) < 0$
- $\lambda_1 > 0 > \lambda_2$: saddle point
  $\Leftarrow det(H) < 0$

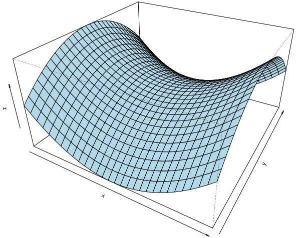
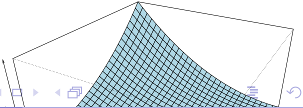



Figure : The function $f(x, y) = x^2 - y^2$.



$\lambda$, is an **eigenvalue** a matrix $A$ if there is a non-zero **x**

# Constrained optimization

To maximize $f(\mathbf{x})$ with respect to $g(\mathbf{x}) = 0$, where both are real-valued, set up the Lagrange function

$$L(\mathbf{x}, \lambda) = f(\mathbf{x}) + \lambda g(\mathbf{x})$$

and solve

$$\frac{\partial L}{\partial x_i} = 0, \ i = 1, \ldots, n$$

along with $g(\mathbf{x}) = 0$.

This will (under certain regularity conditions) give the extrema of $f$ with respect to $g = 0$.

**Example:** Consider the optimization problem to minimize $f(x, y) = x^2 + y^2$ subject to $g(x, y) = x + y - 1 = 0$.

Here the Lagrangian is

$$L(x, y, \lambda) = x^2 + y^2 + \lambda(x + y - 1)$$

and hence

$$0 = \frac{\partial L}{\partial x} = 2x + \lambda \Rightarrow \lambda = -2x$$

$$0 = \frac{\partial L}{\partial y} = 2y + \lambda \Rightarrow \lambda = -2y$$

# Classification of constrained extrema

Write $L(\mathbf{x}, \lambda) = f(\mathbf{x}) + \lambda g(\mathbf{x})$ and suppose $\mathbf{x}^*$ is a potential extremum with $0 = \nabla_{\mathbf{x}^*} L = \nabla f(\mathbf{x}^*) + \lambda^* \nabla g(\mathbf{x}^*)$ and $g(\mathbf{x}^* = 0$.
Further, define the Hessian of $L$, with respect to $\mathbf{x}$ as

$$H = \nabla^2_{\mathbf{x}^*} L = \nabla^2 f(\mathbf{x}^*) + \lambda^* \nabla^2 g(\mathbf{x}^*)$$

If eigenvalues of $H$ are all positive, then $\mathbf{x}^*$ is a local minimum.
**Example:** For $f(x, y) = x^2 + y^2$ and $g(x, y) = x + y - 1$ we have $L(x, y, \lambda) = x^2 + y^2 + \lambda(x + y - 1)$, $\nabla_{\mathbf{x}} L = (2x + \lambda, 2y + \lambda)'$ and thus

$$\nabla^2_{\mathbf{x}} L = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

which has both eigenvalues equal to two and therefore both positive.