Test of hypothesis, P values and related concepts

math612.0 A1: From numbers through algebra to calculus and linear algebra

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The principle of the hypothesis test

The principle is to formulate a hypothesis and an alternative hypothesis, H_0 and H_a respectively, and then select a statistic with a given distribution when H_0 is true and select a rejection region which has a specified probability (α) when H_0 is true.
The rejection region is chosen to reflect H_a, i.e to ensure a high probability of rejection when H_a is true.

The one sided z test for normal mean

Consider testing

$$H_0 : \mu = \mu_0$$

vs

$$H_a : \mu > \mu_0$$

Where data $x_1 \dots x_n$ are collected as independent observations of $X_1 \dots X_n \sim n(\mu, \sigma^2)$ and σ^2 is known. If H_0 is true, then

$$\bar{x} \sim n(\mu_0, \frac{\sigma^2}{n})$$

So,

$$Z = \frac{\bar{x} - \mu_0}{\frac{\sigma}{\sqrt{n}}} \sim n(0, 1)$$

The two-sided z test for a normal mean

$$z := \frac{\overline{x} - \mu_0}{s\sqrt{n}} \sim n(0,1)$$

The one-sided t-test for a single normal mean

Recall that if $X_1, ..., X_n \sim N(\mu, \sigma^2)$ i.i.d. then

$$\frac{\overline{X} - \mu}{S/\sqrt{n}} \sim t_{n-1}$$

Comparing means from normal populations

Suppose data are gathered independently from two normal populations resulting in
$x_1, ..., x_n$ and $y_1, ...y_m$

Comparing means from large samples <ÓI.B.M.>

If $X_1, ...X_n$ and $Y_1, ...Y_m$, are all independent (with finite variance) with expected values of μ_1 and μ_2 respectively, and variances of σ_1^2,and σ_2^2 respectively, then

$$\frac{\overline{X} - \overline{Y} - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^2}{n} + \frac{\sigma_2^2}{m}}} \dot{\sim} n(0,1)$$

if the sample sizes are large enough.

This is the central limit theorem.

The P-value

The p-value of a test is an evaluation of the probability of obtaining results which are as extreme as those observed in the context of the hypothesis.

The concept of significance