# Power and sample sizes

## math612.0 A1: From numbers through algebra to calculus and linear algebra

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# The power of a test

Suppose we have a method to test a null hypothesis against an alternative hypothesis. The test would be "controlled" at some level $\alpha$, i.e. $P[reject\ H_0] \leq \alpha$ whenever $H_0$ is true.

On the other hand, when $H_0$ is false one wants $P[reject\ H_0]$ to be as high as possible.

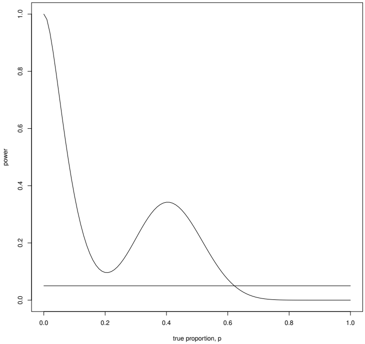
If the parameter to be tested is $\theta$ and $\theta_0$ is a value within $H_0$ and $\theta_a$ is in $H_a$ then we want $P_{\theta_0}[reject\ H_0] \leq \alpha$ and $P_{\theta_a}[reject\ H_0]$ as large as possible.

For a general $\theta$ we write

$$\beta(\theta) = P_\theta[reject\ H_0]$$

for the power of the test

# The power of tests for proportions



# The Power of the one sided z test for the mean

The one sided z-test for the mean ($\mu$) is based on a random sample where $X_1 \dots X_n \sim n(\mu, \sigma^2)$ are independent and $\sigma^2$ is known.

The power of the test for an arbitrary $\mu$ can be computed as:

$$\beta(\mu) = 1 - \Phi\left(\frac{\mu_0 - \mu}{\frac{\sigma}{\sqrt{n}}} + z_{1-\alpha}\right)$$

# Power and sample size for the one-sided z-test for a single normal mean

Suppose we want to test $H_0 : \mu = \mu_0$ vs $H_a : \mu > \mu_0$. We will reject $H_0$ if the observed value

$$z = \frac{\overline{x} - \mu_0}{\sigma/\sqrt{n}}$$

is such that $z > z_{1-\alpha}$.

# The non central t - distribution

Recall that if $Z \sim n(0,1)$ and $U \sim \chi^2{}_v$ are independent then

$$\frac{Z}{\sqrt{\frac{U}{v}}} \sim t_v$$

and it follows for a random sample $X_1 \ldots X_n \sim n(\mu, \sigma^2)$ independent; that

$$\frac{\bar{X}-\mu}{\frac{s}{\sqrt{n}}} = \frac{\frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}}}{\sqrt{\frac{\sum(X_i-\bar{X})^2}{\frac{\sigma^2}{n-1}}}} \sim t_{n-1}$$

# The power of t-test for a normal mean (warning: errors)

# Power and sample size for the one sided t-test for a mean

Suppose we want to calculate the power of a one sided t-test for a single mean (one sample), this can easily be done in R with the `power.t.test` command.

# The power of the 2-sided t-test

A power analysis on a two-sided t-test can be done in R using the *power.t.test* command.

# The power of the 2-sample one and two-sided t-tests

The power of a two sample, one-sided t-test can be computed as follows:

$$\beta_{(\mu_1 \mu_2)} = P_{\mu_1 \mu_2}\left[\frac{Z+\Delta}{\sqrt{U/(n+m-2)}} > t^*_{1-\alpha, n+m-2}\right]$$

and the power of a two sample, two-sided t-test is give by:

$$\beta_{(\mu_1 \mu_2)} = P_{\mu_1 \mu_2}\left[\frac{Z+\Delta}{\sqrt{U/(n+m-2)}} > t^*_{1-\alpha, n+m-2}\right] + P_{\mu_1 \mu_2}\left[\frac{Z+\Delta}{\sqrt{U/(n+m-}}\right.$$

where $\Delta = \frac{(\mu_1 - \mu_2)}{\sigma\sqrt{\frac{1}{n}+\frac{1}{m}}}$ and $U$ is the SSE.

# Sample sizes for two-sample one and two-sided t-tests

The sample size should always satisfy the desired power.

# A case study in power

Want to compute power in analysis of covariance:

$$y_{ij} = \mu_i + \beta x_{ij} + \epsilon_{ij}, \; i = 1, 2, \; j = 1, \ldots J,$$

where $\epsilon_{ij} \sim n(0, \sigma^2)$ are i.i.d.?

This can be done by simulation and can easily be expanded to other cases.