Vectors and Matrix Operations

math612.0 A1: From numbers through algebra to calculus and linear algebra

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Numbers, vectors, matrices

Recall that the set of real numbers is $\mathbb{R}$ and that a vector , $v \in \mathbb{R}^n$ is just an n-tuple of numbers.

Similarly, an *nxm* matrix is just a table of numbers, with n rows and m columns and we can write

$$A_{mn} \in \mathbb{R}^{mn}$$

Note that a vector is normally considered equivalent to a $n \times 1$ matrix i.e. we view these as column vectors.

Elementary Operations

We can define multiplication of a real number k and a vector $v = (v_1, \ldots, v_n)$ by $k \cdot v = (kv_1, \ldots, kv_n)$. The sum of two vectors in $\mathbb{R}^n$, $v = (v_1, \ldots, v_n)$ and $u = (u_1, \ldots, u_n)$ as the vector $v + u = (v_1 + u_1, \ldots, v_n + u_n)$. We can define multiplication of a number and a matrix and the sum of two matrices (of the same sizes) similarly.

The tranpose of a matrix

In R, matrices may be constructed using the "matrix" function and the transpose of A, A', may be obtained in R by using the "t" function:

```
A<-matrix(1:6, nrow=3)
t(A)
```

Matrix multiplication

Matrices A and B can be multiplied together if A is an $n \times p$ matrix and B is an $p \times m$ matrix. The general element $c_i j$ of $n \times m$; $C = AB$ is found by pairing the $i^t h$ row of C with the $j^t h$ column of B, and computing the sum of products of the paired terms.

$$\underbrace{\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}}_{3\times 2} \underbrace{\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}}_{2\times 2} = \begin{bmatrix} 1\cdot 1+2\cdot 1 & 1\cdot 2+2\cdot 3 \\ 3\cdot 1+4\cdot 1 & 3\cdot 2+4\cdot 3 \\ 5\cdot 1+6\cdot 1 & 5\cdot 2+6\cdot 3 \end{bmatrix} = \underbrace{\begin{bmatrix} 3 & 8 \\ 7 & 18 \\ 11 & 28 \end{bmatrix}}_{3\times 2}$$

More on matrix multiplication

Let A, B, and C be $m \times n$, $n \times l$, and $l \times p$ matrices, respectively. Then we have

$$(AB)C = A(BC).$$

In general, matrix multiplication is not commutative, that is $AB \neq BA$. We also have

$$(AB)' = B'A'.$$

In particular, $(Av)'(Av) = v'A'Av$, when v is a $n \times 1$ column vector.

More obvious are the rules

1. $A + (B + C) = (A + B) + C$
2. k(A+B)=kA+kB
3. A(B+C)=AB+AC,

where $k \in \mathbb{R}$ and when the dimensions of the matrices fit.

Linear equations

The unit matrix

The n x n matrix

$$I = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \dots & 0 & 1 \end{bmatrix}$$

is the identity matrix. This is because if a matrix A is n x n then $AI = A$ and $IA = A$

The inverse of a matrix

If A is an $n \times n$ matrix and B is a matrix such that

$$BA = AB = I$$

Then B is said to be the inverse of A, written

$$B = A^{-1}$$

Note that if A is an $n \times n$ matrix for which an inverse exists, then the equation $Ax = b$ can be solved and the solution is $x = A^{-1}b$.