Some notes on matrices and linear operators

math612.0 A1: From numbers through algebra to calculus and linear algebra

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The matrix as a linear operator

Let A be an $m \times n$ matrix, the function

$$T_A : \mathbb{R}^n \to \mathbb{R}^m, T_A(\underline{x}) = A\underline{x},$$

is linear, that is

$$T_A(a\underline{x} + b\underline{y}) = aT_A(\underline{x}) + bT_A(\underline{y})$$

if $\underline{x}, \underline{y} \in \mathbb{R}^n$ and $a, b \in \mathbb{R}$.

Inner products and norms

Assuming x and y are vectors, then we define their inner product by

$$x \cdot y = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$$

where $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ and $y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$

Orthogonal vectors

Two vectors x and y are said to be orthogonal if $x \cdot y = 0$ denoted $x \perp y$

Linear combinations of i.i.d. random variables

Suppose $X_1, ..., X_n$ are i.i.d. random variables and have mean $\mu_1, ..., \mu_n$ and variance σ^2 then the expected value of Y of the linear combination is

$$Y = \sum a_i X_i$$

and if $a_1, ..., a_n$ are real constants then the mean is:

$$\mu_Y = \sum a_i \mu_i$$

and the variance is:

$$\sigma^2 = \sum a_i^2 \sigma_i^2$$

Covariance between linear combinations of i.i.d random variables

Suppose $Y_1, \ldots, Y_n$ are i.i.d., each with mean μ and variance σ^2 and $a, b \in \mathbb{R}^n$. Writing $Y = \begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix}$, consider the linear combination $a'Y$ and $b'Y$.

Random vectors

$Y = (Y_1, \ldots, Y_n)$ is a random vector if $Y_1, \ldots, Y_n$ are random variables.

Transforming random vectors

Suppose

$$\mathbf{Y} = \begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix}$$

is a random vector with $E\mathbf{Y} = \mu$ and $V\mathbf{Y} = \Sigma$ where the variance-covariance matrix

$$\Sigma = \sigma^2 \mathbf{I}$$