Multivariate calculus

math612.0 A1: From numbers through algebra to calculus and linear algebra

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Vector functions of several variables

A vector-valued function of several variables is a function

$$f : \mathbb{R}^m \to \mathbb{R}^n$$

i.e. a function of m dimensional vectors, which returns n dimensional vectors.

The gradient

Suppose the real valued function $f : \mathbb{R}^m \to \mathbb{R}$ is differentiable in each coordinate. Then the gradient of f, denoted ∇f is given by

$$\nabla f(x) = \left(\frac{\partial f}{\partial x_1}, \quad \dots \quad , \frac{\partial f}{\partial x_1} \right).$$

The Jacobian

Now consider a function $f : \mathbb{R}^m \to \mathbb{R}^n$. Write f_i for the i^{th} coordinate of f, so we can write $f(x) = (f_1(x), f_2(x), \dots, f_n(x))$, where $x \in \mathbb{R}^m$. If each coordinate function f_i is differentiable in each variable we can form the *Jacobian matrix* of f:

$$\begin{pmatrix} \nabla f_1 \\ \vdots \\ \nabla f_n \end{pmatrix}.$$

Univariate integration by substitution

If f is a continuous function and g is strictly increasing and differentiable then,

$$\int_{g(a)}^{g(b)} f(x)dx = \int_a^b f(g(t))g'(t)dt$$

Multivariate integration by substitution

Suppose f is a continuous function $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}^n$ is a one-to-one function with continuous partial derivatives. Then if $U \subseteq \mathbb{R}^n$ is a subset,

$$\int_{g(U)} f(\mathrm{x})d\mathrm{x} = \int_U (g(\mathrm{y}))|J|d\mathrm{y}$$

where J is the Jacobian matrix and |J| is the absolute value of it's determinant.

$$J = \left| \begin{bmatrix} \frac{\partial g_1}{\partial y_1} & \frac{\partial g_1}{\partial y_2} & \cdots & \frac{\partial g_1}{\partial y_n} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial g_n}{\partial y_1} & \frac{\partial g_n}{\partial y_2} & \cdots & \frac{\partial g_n}{\partial y_n} \end{bmatrix} \right| = \left| \begin{bmatrix} \nabla g_1 \\ \vdots \\ \nabla g_n \end{bmatrix} \right|$$