Independence, expectations and the moment generating function

math612.0 A1: From numbers through algebra to calculus and linear algebra

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Independent random variables

Recall that two events, A and B, are independent if,

$$P[A \cap B] = P[A]P[B]$$

Since the conditional probability of A given B is defined by:

$$P[A|B] = \frac{P[A \cap B]}{P[B]}$$

We see that A and B are independent if and only if

$$P[A|B] = P[A](when P[B] > 0)$$

Two continuous random variables, X and Y, are similarly independent if,

$$P[X \in A, Y \in B] = P[X \in A]P[Y \in B]$$

Independence and expected values

If X and Y are independent random variables then $E[XY] = E[X]E[Y]$.

Further, if X and Y are independent random variables then $E[g(X)h(Y)] = E[g(X)]E[h(Y)]$ is true if g and h are functions in which expectations exist.

Independence and the covariance

If X and Y are independent then $Cov(X, Y) = 0$.

In fact, if X and Y are independent then $Cov(h(X), g(Y)) = 0$ for any functions g and h in which expected values exist.

The moment generating function

If X is a random variable we define the moment generating function when t exists as: $M(t) := E(e^{tX})$.

Moments and the moment generating function

If $M_X(t)$ is the moment generating function (mgf) of X, then $M_X^{(n)}(0) = E[X^n]$.

The moment generating function of a sum of random variables

$M_{X+Y}(t) = M_X(t) \cdot M_Y(t)$ if X and Y are independent.

Uniqueness of the moment generating function

Moment generating functions (m.g.f.) uniquely determine the probability distribution function for random variables. Thus, if two random variables have the same m.g.f, then they must also have the same distribution.