The gamma distribution

math612.0 A1: From numbers through algebra to calculus and linear algebra

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The gamma distribution

If a random variable X has the density

$$f(x) = \frac{x^{\alpha-1} e^{\frac{-x}{\beta}}}{\Gamma(\alpha)\beta^{\alpha}}$$

where $x > 0$ for some constants α, $\beta > 0$, then X is said to have a gamma distribution.

The mean, variance and mgf of the gamma distribution

Suppose $X \sim G(\alpha, \beta)$ i.e. X has density

$$f(x) = \frac{x^{\alpha-1} e^{-x/\beta}}{\Gamma(\alpha)\beta^{\alpha}}, x > 0$$

Then,

$$E[X] = \alpha\beta$$

$$M(t) = (1 - \beta t)^{-\alpha}$$

$$V[X] = \alpha\beta^2$$

Special cases of the gamma distribution: The exponential and chi-squared distributions

Consider the gamma density,

$$f(x) = \frac{x^{\alpha-1} e^{\frac{-x}{\beta}}}{\Gamma(\alpha)\beta^{\alpha}}, x > 0$$

For parameters $\alpha, \beta > 0$.

If $\alpha = 1$ then

$$f(x) = \frac{1}{\beta} e^{\frac{-x}{\beta}}, x > 0$$

and this is the density of exponential distribution.

Consider next the case $\alpha = \frac{v}{2}$ and $\beta = 2$ where v is an integer, so the density becomes,

The sum of gamma variables

In the general case if $X_1 \dots X_n \sim G(\alpha, \beta)$ are i.i.d. then $X_1 + X_2 + \dots X_n \sim G(n\alpha, \beta)$.

In particular, if $X_1, X_2, \dots, X_v \sim \chi^2$ i.i.d. then $\sum_{i=1}^{v} X_i \sim \chi^2_v$.