math612.2 612.2 Elements of calculus

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`http://mareframe-fp7.org`
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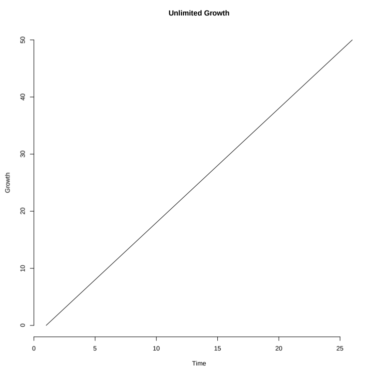
1 Continuity and limits

1.1 The concept of continuity

A function is continuous if it has no jumps. Thus, small changes in each x_0, the input, correspond to small changes in the output, $f(x_0)$.



The above figure is an example of linear growth. Thomas Robert Malthus (1766-1834) warned about the dangers of uninhibited population growth.

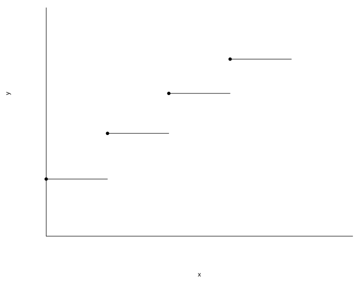
1.1.1 Details

A function is said to be discontinuous if it has jumps. The function is continuous if it has no jumps. Thus, for a continuous function, small changes in each x_0, the input, correspond to small changes in the output, $f(x_0)$.

Note 1.1. Note that polynomials are continuous as are logarithms (for positive numbers).

1.2 Discrete probabilities and cumulative distribution functions

The cumulative distribution function for a discrete random variable is discontinuous.



1.2.1 Details

Definition 1.1. If X is a random variable with a discrete probability distribution and the probability mass function of

$$p(x) = P[X = x]$$

then the **cumulative distribution function**, defined by

$$F(X) = P[X \leq x]$$

is discontinuous, i.e. it jumps at points in which a positive probability occurs.

Note 1.2. When drawing discontinuous functions it is common practice to use a filled circle at $(x, f(x))$ to clarify what the function value is at a point x of discontinuity.

1.2.2 Examples

Example 1.1. If a coin is tossed 3 independent times and X denotes the number of heads, then X can only take on the values 0, 1, 2 and 3. The probability of landing exactly x heads, $P(X = x)$, is $p(x) = \binom{n}{x} p^n (1-p)^{n-x}$. The probabilities are

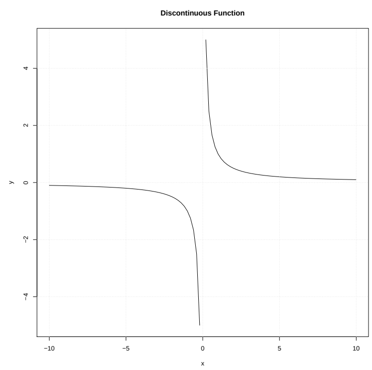
x	p(x)	F(x)
0	1/8	1/8
1	3/8	4/8
2	3/8	7/8
3	1/8	1

The cumulative distribution function, $F(x) = P[X \leq x] = \sum_{t \leq x} p(t)$ has jumps and is therefore discontinuous.

Note 1.3. Notice on the above figure how the circles are filled in, the solid circles indicate where the function value is.

1.3 Notes on discontinuous function

A function is discontinuous for values or ranges of the variable that do not vary continuously as the variable increases. In other words, breaks or jumps.



$f(x) = \frac{1}{x}$, where $x \neq 0$

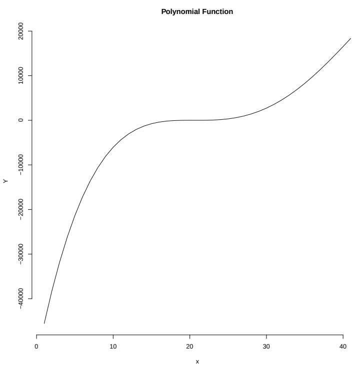
1.3.1 Details

A function can be discontinuous in a number of different ways. Most commonly, it may jump at certain points or increase without bound in certain places.

Consider the function f, defined by $f(x) = 1/x$ when $x \neq 0$. Naturally, $1/x$ is not defined for $x = 0$. This function increases towards $+\infty$ as x goes to zero from the right but decreases to $-\infty$ as x goes to zero from the left. Since the function does not have the same limit from the right and the left, it can not be made continuous at $x = 0$ even if one tries to define $f(0)$ as some number.

1.4 Continuity of polynomials

All polynomials, $p(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$, are continuous.



1.4.1 Details

It is easy to show that simple polynomials such as $p(x) = x$, $p(x) = a + bx$, $p(x) = ax^2 + bx + c$ are continuous functions.

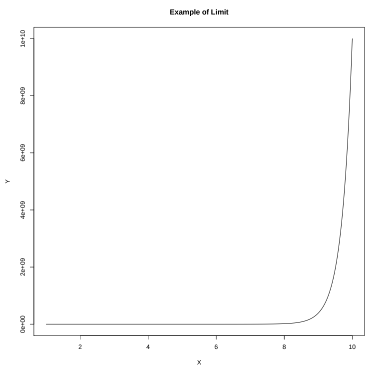
It is generally true that a polynomial of the form

$$p(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$$

is a continuous function.

1.5 Simple Limits

A "limit"is used to describe the value that a function or sequence "approaches"as the input or index approaches some value. Limits are used to define continuity, derivatives and integrals.



$f(x) = x^x$, for $x > 0$

1.5.1 Details

Definition 1.2. A **limit** describes the value that a function or sequence approaches as the input or index approaches some value.

Limits are essential to calculus (and mathematical analysis in general) and are used to define continuity, derivatives and integrals.

Consider a function and a point x_0. If $f(x)$ gets steadily closer to some number c as x gets closer to a number x_0, then c is called the limit of $f(x)$ as x goes to x_0 and is written as:

$$c = \lim_{x \to x_0} f(x)$$

If $c = f(x_0)$ then f is **continuous** at x_0.

1.5.2 Examples

Example 1.2. A simple example of limits:
Evaluate the limit of $f(x) = \frac{x^2-16}{x-4}$ when $x \to 4$, or

$$\lim_{x\to 4} \frac{x^2-16}{x-4}.$$

Notice that in principle we can not simply stick in the value $x = 4$ since we would then get $0/0$ which is not defined. However we can look at the numerator and try to factor it. This gives us:

$$\frac{x^2-16}{x-4} = \frac{(x-4)(x+4)}{x-4} = x+4$$

and the result has the obvious limit of $4+4 = 8$ as $x \to 4$.

Example 1.3. Consider the function

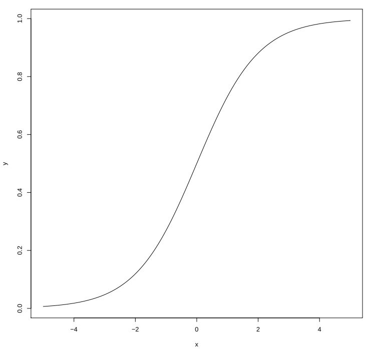
$$g(x) = \frac{1}{x}$$

where x is a positive real number. As x increases, $g(x)$ decreases, approaching 0 but never getting there since $\frac{1}{x} = 0$ has no solution. One can therefore say, "The limit of $g(x)$, as x approaches infinity, is 0," and write

$$\lim_{x\to\infty} g(x) = 0.$$

1.6 More on limits

Limits impose a certain range of values that may be applied to the function.



The function $f(x) = \frac{1}{1+e^{-x}}$.

1.6.1 Examples

Example 1.4. The Beverton-Holt stock recruitment curve is given by:

$$R = \frac{\alpha S}{1+\frac{S}{K}}$$

where $\alpha, K > 0$ are constants and S = biomass and R= recruitment.

The behavior of this curve as S increases $S \to \infty$ is

$$\lim_{S\to\infty} \frac{\alpha S}{1+\frac{S}{K}} = \alpha K.$$

This is seen by rewriting the formula as follows:

$$\lim_{S\to\infty} \frac{\alpha S}{1+\frac{S}{K}} = \lim_{S\to\infty} \frac{\alpha}{\frac{1}{S}+\frac{1}{K}} = \alpha K.$$

Example 1.5. A popular model for proportions is:

$$f(x) = \frac{1}{1+e^{-x}}$$

As x increases, e^{-x} decreases which implies that the term $1+e^{-x}$ decreases and hence $\frac{1}{1+e^{-x}}$ increases, from which it follows that f is an increasing function.
Notice that $f(0) = \frac{1}{2}$ and further,

$$\lim_{x\to\infty} f(x) = 1.$$

This is seen from considering the components:
Since $e^{-x} = \frac{1}{e^x}$ and the exponential function goes to infinity as $x \to \infty$, e^{-x} goes to 0 and hence $f(x)$ goes to 1.

Through a similar analysis one finds that

$$\lim_{x\to-\infty} f(x) = 0,$$

since, as $x \to \infty$, first $-x \to \infty$ and second $e^{-x} \to \infty$.

Example 1.6. Evaluate the limit of

$$f(x) = \frac{\sqrt{x+4}-2}{x}$$

as

$$x \to 0$$

$$\lim_{x\to 0} \frac{\sqrt{x+4}-2}{x}$$

Since the square root is present we cannot just direct substitute the 0 as x. This will give us $\frac{0}{0}$, which is an indeterminate form. We must perform some algebra first. The way to get rid of the radical is to multiply the numerator by the conjugate.

$$\frac{\sqrt{x+4}-2}{x} \cdot \frac{\sqrt{x+4}+2}{\sqrt{x+4}+2}$$

This gives us

$$\frac{(\sqrt{x+4})^2+2(\sqrt{x+4})-2(\sqrt{x+4})-4}{x(\sqrt{x+4}+2)}$$

The numerator reduces to x, and the xs will cancel out leaving us with

$$\frac{1}{\sqrt{x+4}+2}$$

At this point we can direct substitute 0 for x, which will give us

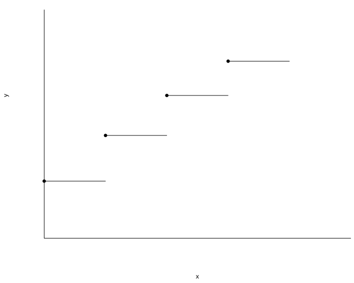
$$\frac{1}{\sqrt{0+4}+2}$$

Therefore,

$$\lim_{x\to 0} \frac{\sqrt{x+4}-2}{x} = \frac{1}{4}$$

1.7 One-sided limits

$f(x)$ may tend towards different numbers depending on whether $x \to x_0$:
from the right ($x \to x_{0+}$)
or from the left ($x \to x_{0-}$).



1.7.1 Details

Sometimes a function is such that $f(x)$ tends to different numbers depending on whether $x \to x_0$ from the right ($x \to x_{0+}$) or from the left ($x \to x_{0-}$).

If

$$\lim_{x\to x_{0+}} f(x) = f(x_0)$$

then we say that f is continuous from the right at x_0.

2 Sequences and series

2.1 Sequences

A **sequence** is a string of indexed numbers $a_1, a_2, a_3, \ldots$. We denote this sequence with $(a_n)_{n\geq 1}$.

2.1.1 Details

In a sequence the same number can appear several times in different places.

2.1.2 Examples

Example 2.1. $(\frac{1}{n})_{n\geq 1}$ is the sequence $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$.

Example 2.2. $(n)_{n\geq 1}$ is the sequence $1, 2, 3, 4, 5, \ldots$.

Example 2.3. $(2^n n)_{n\geq 1}$ is the sequence $2, 8, 24, 64, \ldots$.

2.2 Convergent sequences

A sequence a_n is said to **converge** to the number b if for every $\varepsilon > 0$ we can find an $N \in \mathbb{N}$ such that $|a_n - b| < \varepsilon$ for all $n \geq N$. We denote this with $\lim_{n\to\infty} a_n = b$ or $a_n \to b$, as $n \to \infty$.

2.2.1 Details

A sequence a_n is said to **converge** to the number b if for every $\varepsilon > 0$ we can find an $N \in \mathbb{N}$ such that $|a_n - b| < \varepsilon$ for all $n \geq N$. We denote this with $\lim_{n\to\infty} a_n = b$ or $a_n \to b$, as $n \to \infty$.
If x is a number then,
$(1 + \frac{x}{n})^n \to e^x$ as $n \to \infty$

2.2.2 Examples

Example 2.4. The sequence $(\frac{1}{n})_{n\geq\infty}$ converges to 0 as $n \to \infty$

Example 2.5. If x is a number then,
$(1 + \frac{x}{n})^n \to e^x$ as $n \to \infty$

2.3 Infinite sums (series)

We are interested in, whether infinite sums of sequences can be defined.

2.3.1 Details

Consider a sequence of numbers, $(a_n)_{n\to\infty}$.
Now define another sequence $(s_n)_{n\to\infty}$, where

$$s_n = \sum_{k=1}^{n} a_k.$$

If $(s_n)_{n\to\infty}$ is convergent to $S = \lim_{n\to\infty} s_n$, then we write

$$S = \sum_{n=1}^{\infty} a_n.$$

2.3.2 Examples

Example 2.6. If

$$a_k = x^k, k = 0, 1, ...$$

then

$$s_n = \sum_{k=0}^{n} x^k = x^0 + x^1 + ... + x^n$$

Note also that

$$xs_n = x(x^0 + x^1 + ... + x^n) = x + x^2 + ... + x^{n+1}$$

We have

$$s_n = 1 + x + x^2 + ... + x^n$$
$$xs_n = x + x^2 + ... + x^n + x^{n+1}$$
$$s_n \breve{} xs_n = 1 - x^{n+1}$$

i.e.

$$s_n(1 - x) = 1 - x^{n+1}$$

and we have

$$s_n = \frac{1 - x^{n+1}}{1 - x}$$

if $x \neq 1$. If $0 < x < 1$ then $x^{n+1} \to 0$ as $n \to \infty$ and we obtain $s_n \to \frac{1}{1-x}$ so $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$.

2.4 The exponential function and the Poisson distribution

The exponential function can be written as a series (infinite sum):

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

The Poisson distribution is defined by the probabilities

$$p(x) = e^{-\lambda} \frac{\lambda^x}{x!} \text{ for } x = 0,\ 1,\ 2,\ \ldots$$

2.4.1 Details

The exponential function can be written as a series (infinite sum):

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Knowing this we can see why the Poisson probabilities

$$p(x) = e^{-\lambda}\frac{\lambda^x}{x!}$$

add to one:

$$\sum_{x=0}^{\infty} p(x) = \sum_{x=0}^{\infty} e^{-\lambda}\frac{\lambda^x}{x!} = e^{-\lambda}\sum_{x=0}^{\infty}\frac{\lambda^x}{x!} = e^{-\lambda}e^{\lambda} = 1.$$

2.5 Relation to expected values

The expected value for the Poisson is given by

$$\begin{aligned} \sum_{x=0}^{\infty} xp(x) &= \sum_{x=0}^{\infty} xe^{-\lambda}\frac{\lambda^x}{x!} \\ &= \lambda \end{aligned}$$

2.5.1 Details

The expected value for the Poisson is given by

$$\begin{aligned} \sum_{x=0}^{\infty} xp(x) &= \sum_{x=0}^{\infty} xe^{-\lambda}\frac{\lambda^x}{x!} \\ &= e^{-\lambda}\sum_{x=1}^{\infty}\frac{x\lambda^x}{x!} \\ &= e^{-\lambda}\sum_{x=1}^{\infty}\frac{\lambda^x}{(x-1)!} \\ &= e^{-\lambda}\lambda\sum_{x=1}^{\infty}\frac{\lambda^{(x-1)}}{(x-1)!} \\ &= e^{-\lambda}\lambda\sum_{x=0}^{\infty}\frac{\lambda^x}{x!} \\ &= e^{-\lambda}\lambda e^{\lambda} \\ &= \lambda \end{aligned}$$

3 Slopes of lines and curves

3.1 The slope of a line

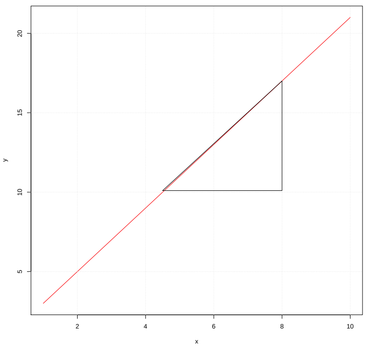
Linear functions produce straight-line graphs. In general, a straight line follows the following equation:

$$y = a + bx,$$

where a and b are fixed numbers.

The line on the graph is the set of points:

$$\{(x,y) : x,y \in \mathbb{R}, y = a + bx\}.$$



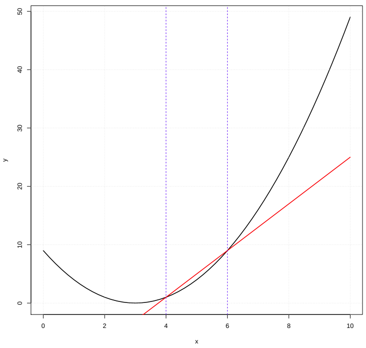
3.1.1 Details

The slope of a straight line represents the change in the y coordinate corresponding to a unit change in the x coordinate.

3.2 Segment slopes

Let's assume we have a more general function $y = f(x)$.

To find the slope of a line segment, consider 2 x-coordinates, x_0 and x_1, and look at the slope between $(x_0, f(x_0))$ and $(x_1, f(x_1))$.



3.2.1 Details

Consider two points, (x_0, y_0) and (x_1, y_1). The slope of the straight line that goes through these points is

$$\frac{y_1 - y_0}{x_1 - x_0}.$$

Thus, the slope of a line segment passing throught the points $(x_0, f(x_0))$ and $(x_1, f(x_1))$, for some function, f, is

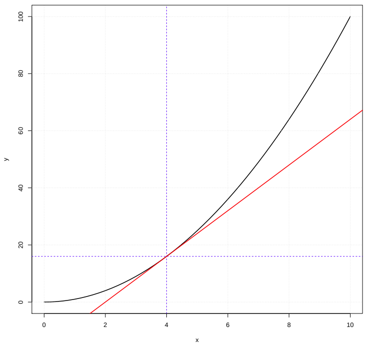
$$\frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

If we let $x_1 = x_0 + h$ then the slope of the segment is

$$\frac{f(x_0 + h) - f(x_0)}{h}.$$

3.3 The slope of $y = x^2$

Consider the task of computing the slope of the function $y = x^2$ at a given point.



3.3.1 Examples

Consider the function $y = f(x) = x^2$.

In order to find the slope at a given point (x_0), we look at

$$y = \frac{f(x_0 + h) - f(x_0)}{h}$$

for small values of h.

For this particular function, $f(x) = x^2$, and hence

$$f(x_0 + h) = (x_0 + h)^2 = x^2 + 2hx_0 + h^2.$$

The slope of a line segment is therefore given by

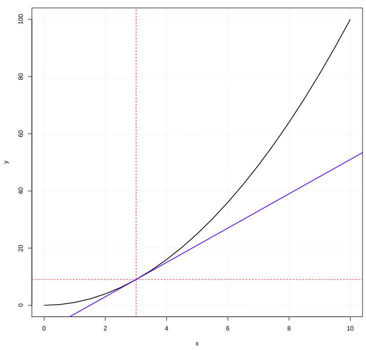
$$\frac{f(x_0 + h) - f(x_0)}{h} = \frac{2hx_0 + h^2}{h} = 2x_0 + h.$$

As we make h steadily smaller, the segment slope, $2x_0 + h$, tends towards $2x_0$. It follows that the slope, y', of the curve *at a general point* x is given by $y' = 2x$.

3.4 The tangent to a curve

A **tangent** to a curve is a line that intersects the curve at exactly one point. The slope of a tangent for the function $y = f(x)$ at the point $(x_0, f(x_0))$ is

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$



3.4.1 Details

To find the slope of the tangent to a curve at a point, we look at the slope of a line segment between the points $(x_0, f(x_0))$ and $(x_0 + h, f(x_0 + h))$, which is

$$\frac{f(x_0 + h) - f(x_0)}{h}$$

and then we take h to be closer and closer to 0. Thus the slope is

$$\lim_{h \to 0} \frac{f(x_0+h)-f(x_0)}{h}$$

when this limit exists.

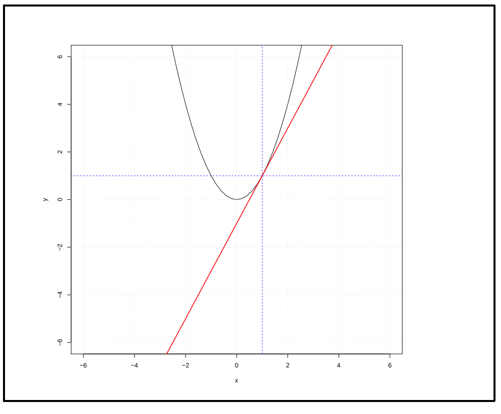
3.4.2 Examples

Example 3.1. We wish to find tangent line for the function $f(x) = x^2$ at the point $(1,1)$. First we need to find the slope of this tangent, it is given as

$$\lim_{h \to 0} \frac{(1+h)^2-1^2}{h} = \lim_{h \to 0} \frac{2h+h^2}{h} = \lim_{h \to 0}(2+h) = 2.$$

Then, since we know the tangent goes through the point $(1,1)$ the line is $y = 2x-1$.

3.5 The slope of a general curve



3.5.1 Details

Imagine a nonlinear function whose graph is a curve described by the equation, $y = f(x)$.

Here we want to find the slope of a line tangent to the curve at a specific point (x_0). The slope of the line segment is given by the equation $\frac{f(x_0+h)-f(x_0)}{h}$.

Reducing h towards zero, gives the slope of this curve if it exists.

4 Derivatives

4.1 The derivative as a limit

The derivative of the function f at the point x is defined as

$$\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$$

if this limit exists.

4.1.1 Details

Definition 4.1. The derivative of the function f at the point x is defined as

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

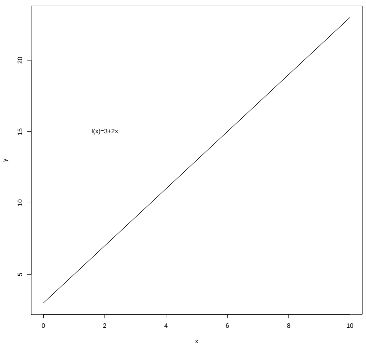
if this limit exists.

When we write $y = f(x)$, we commonly use the notation $\frac{dy}{dx}$ or $f'(x)$ for this limit.

4.2 The derivative of $f(x) = a + bx$

If $f(x) = a + bx$ then $f(x+h) = a + b(x+h) = a + bx + bh$ and thus

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{bh}{h} = b$$



4.2.1 Details

If $f(x) = a + bx$ then $f(x+h) = a + b(x+h) = a + bx + bh$ and thus

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{bh}{h} = b.$$

Thus $f'(x) = b$.

4.3 The derivative of $f(x) = x^n$

If $f(x) = x^n$, then $f'(x) = nx^{n-1}$.

4.3.1 Details

Let $f(x) = x^n$, where n is a positive integer. To calculate f' we use the binomial theorem in the third step:

$$\begin{aligned}
\frac{f(x+h) - f(x)}{h} &= \frac{(x+h)^n - x^n}{h} \\
&= \frac{\sum_{q=0}^{n-1} \binom{n}{q} x^q h^{n-q}}{h} \\
&= \sum_{q=0}^{n-1} \binom{n}{q} x^q h^{n-q-1} \to \binom{n}{n-1} x^{n-1} = nx^{n-1}
\end{aligned}$$

Thus, we obtain $f'(x) = nx^{n-1}$.

4.4 The derivative of ln and exp

If

$$f(x) = e^x$$

then

$$f'(x) = e^x$$

If

$$g(x) = \ln(x)$$

then

$$g'(x) = \frac{1}{x}$$

4.4.1 Details

The derivatives of the exponential function is the exponential function itself i.e.
if

$$f(x) = e^x$$

then

$$f'(x) = e^x$$

The derivatives of the natural logarithm, $\ln(x)$, is $\frac{1}{x}$, i.e. if

$$g(x) = \ln(x)$$

then

$$g'(x) = \frac{1}{x}$$

4.5 The derivative of a sum and linear combination

If f and g are functions then the derivative of $f+g$ is given by $f'+g'$.

4.5.1 Details

Similarly, the derivative of a linear combination is the linear combination of the derivatives. If f and g are functions and $k(x) = af(x)+bg(x)$ then $k'(x) = af'(x)+bg'(x)$.

4.5.2 Examples

Example 4.1. If $f(x) = 2+3x$ and $g(x)+x^3$
then we know that
$f'(x) = 3$, $g(x) = 3x^2$ and if we write

$$h(x) = f(x)+g(x) = 2+3x+x^3$$

then

$$h'(x) = 3+3x^2$$

4.6 The derivative of a polynomial

The derivative of a polynomial is the sum of the derivatives of the terms of the polynomial.

4.6.1 Details

If
$p(x) = a_0 + a_1 x + \ldots + a_n x^n$
then
$p'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \ldots + n a_n x^{(n-1)}$

4.6.2 Examples

Example 4.2. If
$p(x) = 2x^4 + x^3$
then
$p'(x) = 2\frac{dx^4}{dx} + \frac{dx^3}{dx} = 2 \cdot 4x^3 + 3x^2 = 8x^3 + 3x^2$

4.7 The derivative of a product

If

$$h(x) = f(x) \cdot g(x)$$

then

$$h'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$

4.7.1 Details

Consider two functions, f and g and their product, h:

$$h(x) = f(x) \cdot g(x).$$

The derivative of the product is given by

$$h'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x).$$

4.7.2 Examples

Example 4.3. Suppose the function f is given by

$$f(x) = xe^x + x^2 \ln x.$$

Then the derivative can be computed step by step as

$$\begin{aligned} f(x) &= \frac{dx}{dx}e^x + x\frac{de^x}{dx} + \frac{dx^2}{dx}\ln x + x^2\frac{d\ln x}{dx} \\ &= 1 \cdot e^x + x \cdot e^x + 2x \cdot \ln x + x^2 \cdot \frac{1}{x} \\ &= e^x(1+x) + 2x\ln x + x \end{aligned}$$

4.8 Derivatives of composite functions

If f and g are functions and $h = f \circ g$ so that

$h(x) = f(g(x))$ then

$h'(x) = \frac{dh(x)}{dx} = f'(g(x))g'(x)$

4.8.1 Examples

Example 4.4. For fixed x consider:

$$\begin{aligned} f(p) &= \ln(p^x(1-p)^{n-x}) \\ &= \ln p^x + \ln(1-p)^{n-x} \\ &= x\ln p + (n-x)\ln(1-p) \end{aligned}$$

$$\begin{aligned} f'(p) &= x\frac{1}{p} + \frac{n-x}{1-p}(-1) \\ &= \frac{x}{p} - \frac{n-x}{1-p} \end{aligned}$$

Example 4.5. $f(b) = (y-bx)^2$ (y, x fixed)

$$\begin{aligned} f'(b) &= 2(y-bx)(-x) \\ &= -2x(y-bx) \\ &= (-2xy) + (2x^2)b \end{aligned}$$

5 Applications of differentiation

5.1 Tracking the sign of the derivative

If f is a function, then the sign of its derivative, f', indicates whether f is increasing ($f' > 0$), decreasing ($f' < 0$), or zero. f' can be zero at points where f has a maximum, minimum, or a saddle point.

5.1.1 Details

If f is a function, then the sign of its derivative, f', indicates whether f is increasing ($f' > 0$), decreasing ($f' < 0$), or zero. f' can be zero at points where f has a maximum, minimum, or a saddle point.

If $f'(x) > 0$ for $x < x_0$, $f'(x_0) = 0$ and $f'(x) < 0$ for $x > x_0$ then f has a maximum at x_0
If $f'(x) < 0$ for $x < x_0$, $f'(x_0) = 0$ and $f'(x) > 0$ for $x > x_0$ then f has a minimum at x_0
If $f'(x) > 0$ for $x < x_0$, $f'(x_0) = 0$ and $f'(x) > 0$ for $x < x_0$ then f has a saddle point at x_0
If $f'(x) < 0$ for $x < x_0$, $f'(x_0) = 0$ and $f'(x) < 0$ for $x < x_0$ then f has a saddle point at x_0

5.1.2 Examples

Example 5.1. If f is a function such that its derivative is given by

$$f'(x) = (x-1)(x-2)(x-3)(x-4),$$

then applying the above criteria for maxima and minima, we see that f has maxima at 1 and 3 and f has minima at 2 and 4.

5.2 Describing extrema using f''

x_0 with $f'(x_0) = 0$ corresponds to a maximum if $f''(x_0) < 0$
x_0 with $f'(x_0) = 0$ corresponds to a minimum if $f''(x_0) > 0$

5.2.1 Details

If $f'(x_0) = 0$ corresponds to a maximum, then the derivative is decreasing and the second derivative can not be positive, (i.e. $f''(x_0) \leq 0$). In particular, if the second derivative is strictly negative, ($f''(x_0) < 0$), then we are assured that the point is indeed a maximum, and not a saddle point.

If $f'(x_0) = 0$ corresponds to a minimum, then the derivative is increasing and the second derivative can not be negative, (i.e. $f''(x_0) \geq 0$).

If the second derivative is zero, then the point may be a saddle point, as happens with $f(x) = x^3$ at $x = 0$.

5.3 The likelihood function

If p is the probability mass function (p.m.f.):

$$p(x) = P[X = x]$$

then the joint probability of obtaining a sequence of outcomes from independent sampling is

$$p(x_1) \cdot p(x_2) \cdot p(x_3) \dots p(x_n)$$

Suppose each probability includes some parameter θ, this is written,

$$p_\theta(x_1), \dots p_\theta(x_n)$$

If the experiment gives $x_1, x_2 \dots, x_n$ we can write the probability as a function of the parameters:

$$L_{\mathbf{x}}(\theta) = p_\theta(x_1), \dots p_\theta(x_n).$$

This is the *l*ikelihood function.

5.3.1 Details

Definition 5.1. Recall that the **probability mass function (p.m.f)** is a function giving the probability of outcomes of an experiment.

We typically denote the p.m.f. by p so $p(x)$ gives the probability of a given outcome, x, of an experiment. The p.m.f. commonly depends on some parameter. We often write,

$$p(x) = P[X = x].$$

If we take a sample of independent measurements, from p, then the joint probability of a given set of numbers is,

$$p(x_1) \cdot p(x_2) \cdot p(x_3) \dots p(x_n)$$

Suppose each probability includes the same parameter θ, then this is typically written,

$$p_\theta(x_1), \dots p_\theta(x_n)$$

Now consider the set of outcomes $x_1, x_2 \dots, x_n$ from the experiment. We can now take the probability of this outcome as a function of the parameters.

Definition 5.2. $L_{\mathbf{x}}(\theta) = p_\theta(x_1), \dots p_\theta(x_n)$
This is the **likelihood function** and we often seek to maximize it to estimate the unknown parameters.

5.3.2 Examples

Example 5.2. Suppose we toss a biased coin n independent times and obtain x heads, we know the probability of obtaining x heads is,

$$\binom{n}{x} p^x (1-p)^{n-x}$$

The parameter of interest is p and the likelihood function is,

$$L(p) = \binom{n}{x} p^x (1-p)^{n-x}$$

If p is unknown we sometimes wish to maximize this function with respect to p in order to estimate the *t*rue probability p.

5.4 Plotting the likelihood

missing slide – want to give a numeric example and plot L

5.4.1 Examples

missing example – want to give a numeric example and plot L

5.5 Maximum likelihood estimation

If L is a likelihood function for a p.m.f. p_θ, then the value $\hat{\theta}$ which gives the maximum of L:

$$L(\hat{\theta}) = \max_{\theta}(L_\theta)$$

is the maximum likelihood estimator (MLE) of θ

5.5.1 Details

Definition 5.3. If L is a likelihood function for a p.m.f. p_θ, then the value $\hat{\theta}$ which gives the maximum of L:

$$L(\hat{\theta}) = \max_{\theta}(L_\theta)$$

is the **maximum likelihood estimator** of θ

5.5.2 Examples

Example 5.3. If x is the number of heads from n independent tosses of a coin, the likelihood function is:

$$L_x(p) = \binom{n}{x} p^x (1-p)^{n-x}$$

Maximizing this is equivalent to maximizing the logarithm of the likelihood, since logarithmic functions are increasing. The log-likelihood can be written as:

$$\ln(L(p)) = \ln\binom{n}{x} + x\ln(p) + (n-x)\ln(1-p).$$

To find possible maxima , we need to differentiate this formula and set the derivative to zero

$$0 = \frac{dl(p)}{dp} = 0 + \frac{x}{p} + \frac{n-x}{1-p}(-1)$$

$$0 = p(1-p)\frac{(x)}{p} - p(1-p)\frac{n-x}{1-p}$$

$$0 = (1-p)x - p(n-x)$$

$$0 = x - px - pn + px = x - pn$$

So,

$$0 = x - pn$$

$$p = \frac{x}{n}$$

is the extreme and so we can write

$$\hat{p} = \frac{x}{n}$$

for the MLE

5.6 Least squares estimation

Least squares: Estimate the parameters θ by minimizing

$$\sum_{i=1}^{n}(y_i - g_i(\theta))^2$$

5.6.1 Details

Suppose we have a model linking data to parameters. In general we are predicting y_i as g_i (θ).
In this case it makes sense to estimate parameters θ by minimizing

$$\sum_{i=1}^{n}(y_i - g_i(\theta))^2.$$

5.6.2 Examples

Example 5.4. One may predict numbers, x_i, as a mean, μ, plus error. Consider the simple model $x_i = \mu + \varepsilon_i$, where μ is an unknown parameter (constant) and ε_i is the error in measurement when obtaining the i'th observations, x_i, $i = 1, \ldots, n$.

A natural method to estimate the parameter is to minimize the squared deviations

$$\min_{\mu} \sum_{i=1}^{n} (x - \mu)^2 .$$

It is not hard to see that the $\hat{\mu}$ that minimizes this is the mean:

$$\hat{\mu} = \overline{x}.$$

Example 5.5. One also commonly predicts data $y_1, \cdots, y_n$ with values on a straight line, i.e. with $\alpha + \beta x_i$, where $x_1, \ldots, x_n$ are fixed numbers.

This leads to the *r*egression problem of finding parameter values for $\hat{\alpha}$ and $\hat{\beta}$ which gives the best fitting straight line in relation to least squares:

$$\min_{\alpha,\beta} \sum (y_i - (\alpha + \beta x_i))^2$$

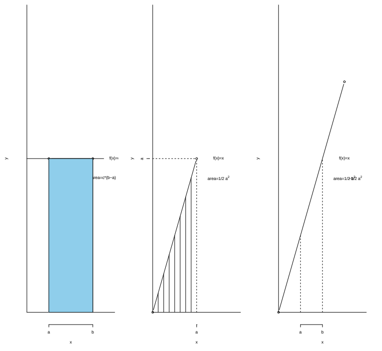
Example 5.6. As a general exercise in finding the extreme of a function, let's look at the function $f(\theta) = \sum_{i=1}^{n} (x_i\theta - 3)^2$ where x_i are some constants. We wish to find the θ that minimizes this sum. We simply differentiate θ to obtain $f'(\theta) = \sum_{i=1}^{n} 2(x_i\theta - 3)x_1 = 2\sum_{i=1}^{n} x_i^2\theta - 2\sum_{i=1}^{n} 3x_i$. Thus,

$$f'(\theta) = 2\theta \sum_{i=1}^{n} x_i^2 - 2\sum_{i=1}^{n} 3x_i = 0$$
$$\Leftrightarrow \theta = \frac{\sum_{i=1}^{n} 3x_i}{\sum_{i=1}^{n} x_i^2}.$$

6 Integrals and probability density functions

6.1 Area under a curve

The area under a curve between x=a and x=b (for a positive function) is called the integral of the function.



Example 1, 2 and 3

6.1.1 Details

Definition 6.1. The area under a curve between x=a and x=b (for a positive function) is called the **integral of the function** and is denoted: $\int_a^b f(x)dx$ when it exists.

6.2 The antiderivative

Given a function f, if there is another function F such that $F' = f$, we say that F is the *antiderivative* of f. For a function f the antiderivative is denoted by $\int f dx$.
Note that if F is one antiderivative of f and C is a constant, then $G = F + C$ is also an antiderivative. It is therefore customary to always include the constant, e.g. $\int x dx = \frac{1}{2}x^2 + C$.

6.2.1 Examples

Example 6.1. The antiderivative of x to a power raises the power. $\int x^n dx = \frac{1}{n+1}x^{n+1} + C$.

Example 6.2. $\int e^x dx = e^x + C$.

Example 6.3. $\int \frac{1}{x} dx = \ln(x) + C$.

Example 6.4. $\int 2xe^{x^2}dx = e^{x^2} + C$.

6.3 The fundamental theorem of calculus

If $F'(x) = f(x)$ for $x \in [a,b]$, then $\int_a^b f(x)dx = F(b) - F(a)$

6.3.1 Detail

It is not too hard to see that the area under the graph of a positive function f on the interval $[a,b]$ must be equal to the difference of the values of its antiderivative at a and b. This also holds for functions which take on negative values and is formally stated below.

Definition 6.2. Fundamental theorem of calculus: If F is the antiderivative of f, i.e. $F' = f$ for $x \in [a,b]$, then $\int_a^b f(x)dx = F(b) - F(a)$.
This difference is often written as $\int_a^b f dx$ or $[F(x)]_a^b$.

6.3.2 Examples

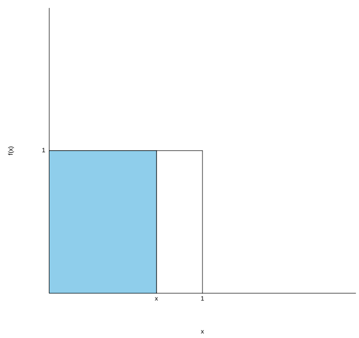
Example 6.5. The area under the graph of x^n between 0 and 3 is $\int_0^3 x^n dx = [\frac{1}{n+1}x^{n+1}]_0^3 = \frac{1}{n+1}3^{n+1} - \frac{1}{n+1}0^{n+1} = \frac{3^{n+1}}{n+1}$

Example 6.6. The area under the graph of e^x between 3 and 4 is $\int_3^4 e^x dx = [e^x]_3^4 = e^4 - e^3$

Example 6.7. The area under the graph of $\frac{1}{x}$ between 1 and a is $\int_1^a \frac{1}{x}dx = [\ln(x)]_1^a = \ln(a) - \ln(1) = \ln(a)$.

6.4 Density functions

The probability density function (p.d.f.) and the cumulative distribution function (c.d.f.).



6.4.1 Details

Definition 6.3. If X is a random variable such that

$$P(a \leq X \leq b) = \int_a^b f(x)dx,$$

for some function f which satisfies $f(x) \geq 0$ for all x and

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

then f is said to be a **probability density function (p.d.f.)** for X.

Definition 6.4. The function

$$F(x) = \int_{-\infty}^{x} f(t)dt$$

is the **cumulative distribution function (c.d.f.)**.

6.4.2 Examples

Example 6.8. Consider a random variable X from the uniform distribution, denoted by $X \sim U(0,1)$. This distribution has density

$$f(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq 1 \\ 0 & \text{e.w.} \end{cases}$$

The cumulative distribution function is given by

$$P[X \leq x] = \int_{-\infty}^{x} f(t)dt = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } 0 \leq x \leq 1 \\ 1 & \end{cases}$$

Example 6.9. Suppose $X \sim P(\lambda)$, where X may denote the number of events per unit time. The p.m.f. of X is described by $p(x) = P[X = x] = e^{-\lambda}\frac{\lambda^x}{x!}$ for $x = 0, 1, 2, \ldots$. Consider now the waiting time, T, between events, or simply until the first event. Consider the event $T > t$ for some number t>0. If $X \sim p(\lambda)$ denotes the number of events per unit time, then let X_t denote the number of events during the time period for 0 through t. Then it is natural to assume

$X_t \sim P(\lambda t)$ and it follows that $T > t$ if and only if $X_t = 0$ and we obtain $P[T > t] = P[X_t = 0] = e^{-\lambda t}$. It follows that the c.d.f. of T is $F_T(t) = P[T \le t] = 1 - P[T > t] = 1 - e^{-\lambda t}$ for $t > 0$.

The p.d.f. of T is therefore $f_T(t) = F_T'(t) = \frac{d}{dt}F_T(t) = \frac{d}{dt}(1 - e^{-\lambda t} = 0 - e^{-\lambda t} * (-\lambda) = \lambda e^{-\lambda t}$ for $t \ge 0$ and $f_T(t) = 0$ for $t = 0$.

The resulting density

$$f(t) = \begin{cases} \lambda e^{-\lambda t} & \text{for} \quad t \ge 0 \\ 0 & \text{for} \quad t < 0 \end{cases}$$

describes the exponential distribution.
This distribution has the expected value

$$E[T] = \int_{-\infty}^{\infty} t f(t) dt = \int_{0}^{\infty} t \lambda e^{-\lambda t} dt.$$

the stuff below is all messed up...
We set $u = \lambda t$ and $du = \lambda dt$ to obtain

$$\int u e^{-u} du = \frac{1}{\lambda} \int_0^{\infty} u e^{-u} du = \frac{1}{\lambda} = \int_0^{\infty} 1 \cdot e^{-u} du$$

$$= \left[-u e^{-u} \right]_0^{\infty}$$

$$= \left[\frac{1}{\lambda}(-e^{-u}) \right]_0^{\infty} - 0 = \frac{1}{\lambda}.$$

6.5 Probabilities in R: The normal distribution

R has functions to compute values of probability density functions (p.d.f.) and cumulative distribution functions (c.m.d.) for most common distributions.

6.5.1 Details

The p.d.f. for the normal distribution is

$$p(t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}$$

The c.d.f. for the normal distribution is

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt$$

6.5.2 Examples

Example 6.10. dnorm() gives the value of the normal p.d.f.

Example 6.11. pnorm() gives the value of the normal c.d.f.

6.6 Some rules of integration

6.6.1 Examples

Example 6.12. Using integration by parts we obtain

$$\int \ln(x)xdx = \frac{1}{2}x^2\ln(x) - \int \frac{1}{2}x^2 \cdot \frac{1}{x}dx = \frac{1}{2}x^2\ln(x) - \int \frac{1}{2}xdx = \frac{1}{2}x^2\ln(x) - \frac{1}{4}x^2.$$

Example 6.13. Consider $\int_1^2 2xe^{x^2}dx$. By setting $x = g(t) = \sqrt{t}$ we obtain

$$\int_1^2 2xe^{x^2}dx = \int_1^4 2\sqrt{t}e^t\frac{1}{2\sqrt{t}}dt = \int_1^4 e^t dt = e^4 - e.$$

6.6.2 Handout

The two most common "tricks"applied in integration are a) integration by parts and b) integration by substitution.

a) **Integration by parts**

$$(fg)' = f'g + fg'$$

by integrating both sides of the equation we obtain:

$$fg = \int f'gdx + \int fg'dx \Leftrightarrow \int fg'dx = fg - \int f'gdx$$

b) **Integration by substitution**

Consider the definite integral $\int_a^b f(x)dx$ and let g be a one-to-one differential function for the interval (c,d) to (a,b). Then

$$\int_a^b f(x)dx = \int_c^d f(g(y))g'(y)dy$$

7 Principles of programming

7.1 Modularity

Modularity involves designing a system that is divided into a set of functional units (named modules) that can be composed into a larger application.

Any programming project should be split into logical module pieces of code which are combined into a complete program.

7.1.1 Details

Typically input, initialization, analysis, and output commands are grouped into separate parts.

7.1.2 Examples

Example 7.1. Input

```
dat<-read.table("http://notendur.hi.is/~gunnar/kennsla/alsm/data/
   set115.dat", header=T)
cols<- c("le", "osl")
```

Analysis

```
Mn<-mean(dat[, cols[1]])
```

Output

```
print (Mn)
```

7.2 Modularity and functions

In many cases groups of commands can be collected together into a function.

7.2.1 Details

Typically a project has several such functions.

7.2.2 Examples

Example 7.2. Suppose you want to plot the weight vs. length for several datasets in

`http://hi.is/~gunnar/kennsla/alsm/data`

A function can then be set up with the file number as an argument:

```
plotwtle<-function (fnum){
fname<-paste(
"http://hi.is/~gunnar/kennsla/alsm/data/set",fnum,".dat",sep="")
 cat("The␣URL␣B", fname,"\n")
```

```
dat<-read.table(fname,header=T)
ttl<-paste("Data from file number", fnum)
plot(dat$le,dat$osl,main=ttl)
}
```

Now call this with

```
plotwtle(105)
```

7.3 Modularity and files

It is advisable to split larger projects into several manageable files.

7.3.1 Details

Once a project reaches more than five lines of code, it should be stored in one or more separate files. In order to combine these files a single "source" command file can be created.

Typically function definitions are stored in separate files, so one may have several separate files like:

"input.r"
"function.r"
"analysis.r"
output.r"

While developing the analysis, the data would only be read once with

source("input.r")

The goal of this practice is to end up with a set of files which are completely self-contained, so one can start with an empty R session and give only the commands like:

source ("input.r")
source ("functions.r")
source ("analysis.r")

Furthermore, this ensures repeatability.

7.3.2 Examples

Example 7.3. For a given project "input", "functions" "analysis" and "output" files can be created as below.
input.r

```
dat<-read.table("http://notendur.hi.is/~gunnar/kennsla/alsm/data/
   set115.dat", header=T)
```

functions.r

```
plotwtle<-function(fnum){
 fname<-paste("http://notendur.hi.is/~gunnar/kennsla/alsm/data/set",
    fnum,".dat",sep="")
  cat("The URL is",fname,"\n")
  dat<-read.table(fname,header=T)
  ttl<-paste("My data set was",fnum)
  plot(dat$le,dat$osl,main=ttl,xlab="Length(cm)",ylab="Live weight (
     g)")
}
```

output.r

```
source("functions.r")
for(i in 101:150){
  fnam<-paste("plot",i,".pdf",sep="")
  pdf(fnam)
  plotwtle(i)
  dev.off()
}
```

These files can be executed with source commands as below:

source (“input.r”)

source (“functions.r”)

source (“output.r”)

7.4 Structuring an R project

7.4.1 Details

We already covered how to split code into different functions and linking them together with the help of one executable file that is "sourcing"the others. However, when you undertake a larger project, there will be a lot of different data and files and it is very advisable to have a consistent structure throughout the project.

A common project layout is to allocate all project files into a folder, something along the lines of:

```
/project
/data
/src
/doc
/figs (or /out)
```

Such a structure is quite normal in programming languages such as C, Matlab, and R.

Purpose of the different folders:

/data: Contains all important data to the project, which you will use. This folder should be read-only! No function is allowed to write anything into this folder.

/src: (abbreviation for "source(-code)") Here you will store all the functions that you programmed. You can decide to store the executable function here as well or, alternatively, have that one in the root project folder.

/doc: Contains further documentation material about your project. This could be, for example, readme files for other people who use your functions, or the paper you wrote about the project, or the latex files while you're writing.

/figs or /out: Here your functions are allowed to write and can produce the different results, like graphs, figures or anything else.

Finally, a large programming project should at some stage be split into packages and stored on dedicated servers such as github or CRAN.

7.4.2 Examples

Example 7.4. Consider first the issue of maintaining the code itself. It is common for R beginners to only work interactively within the command-line interface. However, it is essential that the code be kept in one or more files.

For large projects these will be several different files, each with its own purpose. To run a complete analysis one would typically set up one file to run all the tasks by reading in data through analyses to outputs.

For example, a file named "run.r"could contain the sequence of commands:

source("setup.r")

source("analysis.r")

source("plot.r")

7.5 Loops, for

If a piece of code is to be run repeatedly, the for-loop is normally used.

7.5.1 Details

If a piece of code is to be run repeatedly, the for-loop is normally used. The R code form is:

```
for(index in sequence){
commands
}
```

7.5.2 Examples

Example 7.5. To add numbers we can use

```
tot <- 100
for(i in 1:100){
  tot <- tot + i
}
cat ("the␣sum␣is␣", tot, "\n")
```

Example 7.6. Define the plot function

```
plotwtle <- AS BEFORE
```

To plot several of these we can use a sequence:

```
plotwtle(101)
plotwtle(102)
.
.
.
```

or a loop

```
for (i in 101:150){
  fname<- paste("plot", i, ".pdf", sep="")
  pdf(fname)
  plotwtle(i)
  dev.off()
}
```

7.6 The if and ifelse commands

The "if"statement is used to conditionally execute statements.
The "ifelse"statement conditionally replaces elements of a structure.

7.6.1 Examples

Example 7.7. If we want to compute x^x for x-values in the range 0 through 5, we can use

```
xlist<-seq(0,5,0.01)
y<-NULL
for(x in xlist){
  if(x==0){
    y<-c(y,1)
```

```
  }else{
    y<-c(y,x**x)
  }
}
```

Example 7.8. `x<-seq(0,5,0.01)`
```
y<-ifelse(x==0,1,x^x)
```

Example 7.9. `dat<-read.table ("file")`
```
dat<-ifelse (dat==0,0.01,dat)
```

Example 7.10. `x<-ifelse (is.na(x),0,x)`

7.7 Indenting

Code should be properly indented!

7.7.1 Details

fFunctions, for-loops, and if-statements should always be indented.

7.8 Comments

All code should contain informative comments. Comments are separated out from code using the pound symbol (#).

7.8.1 Examples

Example 7.11. ####################

```
####SETUP DATA####
####################

dat<-read.table(filename)
x<-log(dat$le) #log-transformation of length
```

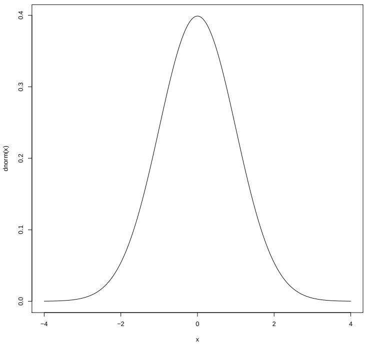
```
y<-log(dat$wt) #log-transformation of weight

######################
####THE ANALYSIS####
######################
```

8 The Central Limit Theorem and related topics

8.1 The Central Limit Theorem

If measurements are obtained independently and come from a process with finite variance, then the distribution of their mean tends towards a Gaussian (normal) distribution as the sample size increases.



The standard normal density

8.1.1 Details

Theorem 8.1 The **Central Limit Theorem** states that if $X_1, X_2, \ldots$ are independent and identically distributed random variables with mean μ and (finite) variance σ^2, then the distribution of $\bar{X}_n := \frac{X_1+\cdots+X_n}{n}$ tends towards a normal distribution.

It follows that for a large enough sample size n, the distribution random variable $\bar{X}_n$ can be approximated by $n(\mu, \sigma^2/n)$.
The standard normal distribution is given by the p.d.f.

$$\varphi(z) = \frac{1}{\sqrt{2\pi}} e^{\frac{-z^2}{2}}$$

for $z \in \mathbb{R}$.

The standard normal distribution has an expected value of zero,

$$\mu = \int z\varphi(z)dz = 0$$

and a variance of

$$\sigma^2 = \int (z-\mu)^2 \varphi(z)dz = 1$$

If a random variable Z has the standard normal (or Gaussian) distribution, we write $Z \sim n(0,1)$.

If we define a new random variable, Y, by writing $Y = \sigma Z + \mu$, then Y has an expected value of μ, a variance of σ^2 and a density (p.d.f.) given by the formula:

$$f(y) = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(y-\mu)^2}{2\sigma^2}}.$$

This is general normal (or Gaussian) density, with mean μ and variance σ^2.
The Central Limit Theorem states that if you take the mean of several independent random variables, the distribution of that mean will look more and more like a Gaussian distribution (if the variance of the original random variables is finite).
More precisely, the cumulative distribution function of

$$\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}}$$

converges to Φ, the $n(0,1)$ cumulative distribution function.

8.1.2 Examples

Example 8.1. If we collect measurements on waiting times, these are typically assumed to come from an exponential distribution with density

$$f(t) = \lambda e^{-\lambda t}, \text{ for } t > 0$$

The Central Limit Theorem states that the mean of several such waiting times will tend to have a normal distribution.

Example 8.2. We are often interested in computing

$$w = \frac{\bar{x} - \mu_0}{\frac{s}{\sqrt{n}}}$$

which comes from a t-distribution (see below), if the x_i are independent outcomes from a normal distribution.
However, if n is large and σ^2 is finite then w values will look as though they came from a normal distribution. This is in part a consequence of the Central Limit Theorem, but also of the fact that s will become close to σ as n increases.

8.2 Properties of the binomial and Poisson distributions

The binomial distribution is really a sum of 0 and 1 values (counts of failures = 0 and successes =1). So, a simple, single binomial outcome will correspond to coming from a normal distribution if the count is large enough.

8.2.1 Details

Consider the binomial probabilities:

$$p(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

for $x = 0, 1, 2, 3, \cdots, n$ where n is a non-negative integer. Suppose p is a small positive number, specifically consider a sequence of decreasing p-values, specified with $p_n = \frac{\lambda}{n}$ and consider the behavior of the probability as $n \to \infty$. We obtain:

$$\begin{aligned}
\binom{n}{x} p_n^x (1-p_n)^{n-x} &= \frac{n!}{x!(n-x!)} \left(\frac{\lambda}{n}\right)^x \left(1-\frac{\lambda}{n}\right)^{n-x} && (1)\\
&= \frac{n(n-1)(n-2)\cdots(n-x+1)}{x!} \frac{\frac{\lambda}{n}^x}{(1-\frac{\lambda}{n})^x} \left(1-\frac{\lambda}{n}\right)^n && (2)\\
&= \frac{n(n-1)(n-2)\cdots(n-x+1)}{x!n^x} \frac{\lambda^x}{(1-\frac{\lambda}{n})^x} \left(1-\frac{\lambda}{n}\right)^n && (3)\\
& && (4)
\end{aligned}$$

Note 8.1. Notice that $\frac{n(n-1)(n-2)\cdots(n-x+1)}{n^x} \to 1$ as $n \to \infty$. Also notice that $(1-\frac{\lambda}{n})^x \to 1$ as $n \to \infty$. Also

$$\lim_{n\to\infty} \left(1-\frac{\lambda}{n}\right) = e^{-\lambda}$$

and it follows that

$$\lim_{n\to\infty} \binom{n}{x} p_n^x (1-p_n)^{n-x} = \frac{e^{-\lambda}\lambda^x}{x!}, x = 0, 1, 2, \cdots, n$$

and hence the binomial probabilities may be approximated with the corresponding Poisson.

8.2.2 Examples

Example 8.3. The mean of a binomial (n,p) variable is $\mu = n \cdot p$ and the variance is $\sigma^2 = np(1-p)$.

The R command $dbinom(q,n,p)$ calculates the probability of q successes in n trials assuming that the probability of a success is p in each trial (binomial distribution), and the R command $pbinom(q,n,p)$ calculates the probability of obtaining q or fewer successes in n trials.
The normal approximation of this distribution can be calculated with $pnorm(q,mu,sigma)$ which becomes $pnorm(q,n*p,sqrt(n*p(1-p))$. Three numerical examples (note that pbinom and pnorm give similar values for large n):

```
pbinom(3,10,0.2)
[1] 0.8791261
pnorm(3,10*0.2,sqrt(10*0.2*(1-0.2)))
[1] 0.7854023
```

```
pbinom(3,20,0.2)
[1] 0.4114489
pnorm(3,20*0.2,sqrt(20*0.2*(1-0.2)))
[1] 0.2880751

pbinom(30,200,0.2)
[1] 0.04302156
pnorm(30,200*0.2,sqrt(200*0.2*(1-0.2)))
[1] 0.03854994
```

Example 8.4. We are often interested in computing $w = \frac{\bar{x}-\mu}{s/\sqrt{n}}$ which has a t-distribution if the x_i are independent outcomes from a normal distribution. If n is large and σ^2 is finite, this will look as if it comes from a normal distribution.

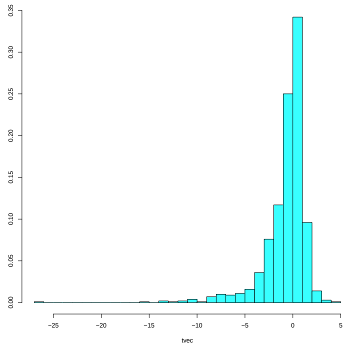
The numerical examples below demonstrate how the t-distribution approaches the normal distribution.

```
qnorm(0.7)
[1] 0.5244005
#This is the value which gives the cumulative probability of p=0.7
   for a n~(0,1)
qt(0.7,2)
[1] 0.6172134
#The value, which gives the cumulative probability of p=0.7 with n=2
    for the t-distribution.
qt(0.7,5)
[1] 0.5594296
qt(0.7,10)
[1] 0.541528
qt(0.7,20)
[1] 0.5328628


qt(0.7,100)
[1] 0.5260763
```

8.3 Monte Carlo simulation

If we know an underlying process we can simulate data from the process and evaluate the distribution of any quantity based on such data.



A simulated set of t-values based on data from an exponential distribution.

8.3.1 Examples

Example 8.5. Suppose our measurements come from an exponential distribution and we want to compute

$$t = \frac{\overline{x} - \mu}{s/\sqrt{n}}$$

but we want to know the distribution of those when μ is the true mean.

For instance, $n = 5$ and $\mu = 1$, we can simulate (repeatedly) $x_1, \ldots, x_5$ and compute a t-value for each. The following R commands can be used for this:

```
library(MASS)
n<-5
mu<-1
lambda<-1
tvec<-NULL
for(sim in 1:10000){
   x<-rexp(n,lambda)
   xbar<-mean(x)
   s<-sd(x)
   t<-(xbar-mu)/(s/sqrt(n))
   tvec<-c(tvec,t)
}

#then do...

 truehist(tvec)   #truehist gives a better histogram
 sort(tvec)[9750]
 sort(tvec)[250]
```

9 Miscellanea

9.1 Simple probabilities in R

R has functions to compute probabilities based on most common distributions.

If X is a random variable with a known distribution, then R can typically compute values of the cumulative distribution function or:

$$F(x) = P[X \leq x]$$

9.1.1 Examples

Example 9.1. If $X \sim b(n, p)$ has binomial distribution, i.e.

$$P(X = x) = \binom{n}{x} p^x (1-p)^{n-x},$$

then cumulative probabilities can be computed with *pbinom*, e.g.

```
pbinom(5,10,0.5)
```

gives

$$P[X \leq 5] = 0.623$$

where

$$X \sim b(n = 10, p = \frac{1}{2}).$$

This can also be computed by hand. Here we have $n = 10$, $p = 1/2$ and the probability $P[X \leq 5]$ is obtained by adding up the individual probabilities, $P[X = 0] + P[X = 1] + P[X = 2] + P[X = 3] + P[X = 4] + P[X = 5]$

$$P[X \leq 5] = \sum_{x=0}^{5} \binom{10}{x} \frac{1}{2}^x \frac{1}{2}^{10-x}.$$

This becomes

$$P[X \leq 5] = \binom{10}{0} \frac{1}{2}^0 \frac{1}{2}^{10-0} + \binom{10}{1} \frac{1}{2}^1 \frac{1}{2}^{10-1} + \binom{10}{1} \frac{1}{2}^2 \frac{1}{2}^{10-2} + \binom{10}{3} \frac{1}{2}^3 \frac{1}{2}^{10-3} + \binom{10}{4} \frac{1}{2}^4 \frac{1}{2}^{10-4} + \binom{10}{5} \frac{1}{2}^5 \frac{1}{2}^{10-}$$

or

$$P[X \leq 5] = \binom{10}{0} \frac{1}{2}^{10} + \binom{10}{1} \frac{1}{2}^{10} + \binom{10}{1} \frac{1}{2}^{10} + \binom{10}{3} \frac{1}{2}^{10} + \binom{10}{4} \frac{1}{2}^{10} + \binom{10}{5} \frac{1}{2}^{10} = \frac{1}{2}^{10} [1 + 10 + 45 + \ldots].$$

Furthermore,

```
pbinom(10,10,0.5)
[1] 1
```

and

```
pbinom(0,10,0.5)
[1] 0.0009765625
```

It is sometimes of interest to compute $P[X = x]$ in this case, and this is given by the *dbinom* function, e.g.

```
dbinom(1,10,0.5)
[1] 0.009765625
```

or $\frac{10}{1024}$

Example 9.2. Suppose X has a uniform distribution between 0 and 1, i.e. $X \sim U(0,1)$. Then the *punif* function will return probabilities of the form

$$P[X \le x] = \int_{-\infty}^{x} f(t)dt = \int_{0}^{x} f(t)dt$$

where $f(t) = 1$ if $0 \le t \le 1$ and $f(t) = 0$. For example:

```
punif(0.75)
[1] 0.75
```

To obtain $P[a \le X \le b]$, we use *punif* twice, e.g.

```
punif(0.75)-punif(0.25)
[1] 0.5
```

9.2 Computing normal probabilities in R

To compute probabilities $X \sim n(\mu,\sigma^2)$ is usually transformed, since we know that

$$Z := \frac{X-\mu}{\sigma} \sim (0,1)$$

The probabilities can then be computed for either X or Z with the *pnorm* function in R.

9.2.1 Details

Suppose X has a normal distribution with mean μ and variance

$$X \sim n(\mu,\sigma^2)$$

then to compute probabilities, X is usually transformed, since we know that

$$Z = \frac{X-\mu}{\sigma} \sim (0,1)$$

and the probabilities can be computed for either X or Z with the *pnorm* function.

9.2.2 Examples

Example 9.3. If $Z \sim n(0,1)$ then we can e.g. obtain $P[Z \leq 1.96]$ with

```
pnorm(1.96)
[1] 0.9750021

pnorm(0)
[1] 0.5

pnorm(1.96)-pnorm(1.96)
[1] 0

pnorm(1.96)-pnorm(-1.96)
[1] 0.9500042
```

The last one gives the area between -1.96 and 1.96.

Example 9.4. If $X \sim n(42,3^2)$ then we can compute probabilites either by transforming

$$P[X \leq x] = P[\frac{X-\mu}{\sigma} \leq \frac{x-\mu}{\sigma}]$$
$$= P[Z \leq \frac{x-\mu}{\sigma}]$$

and calling *pnorm* with the computed value $z = \frac{x-\mu}{\sigma}$, or call *pnorm* with x and specify μ and σ.

To compute $P[X \leq 48]$, either set $z = (48-42)/3 = 2$ and obtain

```
pnorm(2)
[1] 0.9772499
```

or specify μ and σ

```
pnorm(42,42,3)
[1] 0.5
```

9.3 Introduction to hypothesis testing

9.3.1 Details

If we have a random sample $x_1,\ldots,x_n$ from a normal distribution, then we consider them to be outcomes of independent random variables $X_1,\ldots,X_n$ where $X_i \sim n(\mu,\sigma^2)$. Typically, μ and σ^2 are unknown but assume for now that σ^2 is known.

Consider the hypothesis:

$H_0 : \mu = \mu_0$ vs. $H_1 : \mu > \mu_0$

where μ_0 is a specified number.

Under the assumption of independence, the sample mean

$$\overline{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

is also an observation from a normal distribution, with mean μ but a smaller variance.Specifically, $\overline{x}$ is the outcome of

$$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

and

$$X \sim n(\mu, \frac{\sigma^2}{n})$$

so the standard deviation of X is $\frac{\sigma}{\sqrt{n}}$, so the appropriate error measure for $\overline{x}$ is $frac\sigma\sqrt{n}$, when σ is unknown.

If H_0 is true, then

$$z := \frac{\overline{x} - \mu_0}{\sigma/\sqrt{n}}$$

is an observation from an $n \sim n(0,1)$ distribution, i.e. an outcome of

$$Z = \frac{\overline{X} - \mu_0}{\sigma/\sqrt{n}}$$

where $Z \sim n(0,1)$ when H_0 is correct. It follows that e.g. $P[|Z| > 1.96] = 0.05$ and if we observe $|Z| > 1.96$ then we reject the null hypothesis.

Note that the value z* = 1.96 is a quantile of the normal distribution and we can obtain other quantiles with the *pnorm* function, e.g. $pnorm(0.975)$ gives 1.96.