Probability theory

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Randomness

- Descriptive statistics describe the sample that we have obtained
- Statistical inference uses the sample to draw conclusions about the whole population.
- The variables that we measure are influenced by some randomness.
- We therefore look at every measurement as a random phenomena.
- In this lecture we look closer at random phenomena.

Events, outcomes and outcome space.

Outcome and outcome space

Every random phenomena has certain possible **outcomes**. The set of all possible outcomes is the **outcome space** an is denoted with Ω.

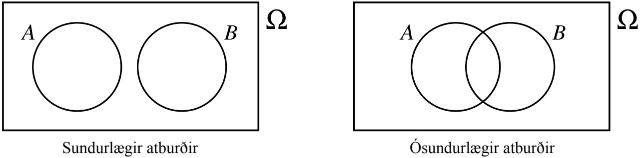
Event

An **Event** is a particular outcome or a set of particular outcomes of a random phenomena.

Disjoint events

Disjoint events

We say that events A and B are **disjoint** if they contain no common outcome.



Mynd: Disjoint and joint events.

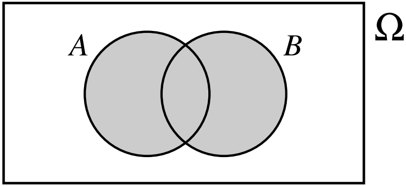
Union and intersection of events

Union of events

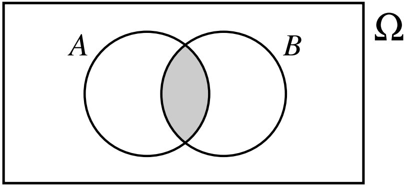
The **union** of events A and B is denoted $A \cup B$. It is the set of all outcomes that are in **either** A or B or **both of them**.

Intersection of events

The **Intersection** of events A and B is denoted $A \cap B$. It is the set of al outcomes that are in **both** A and B. If A and B are disjoint, then their intersection is **empty**.



$A \cup B$

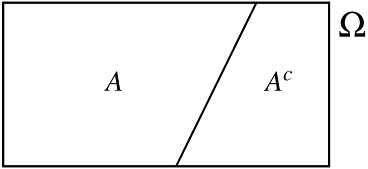


$A \cap B$

The complement of an event

Complement of an event

The **Complement** of an event A is denoted A^C. It s the set of all outcomes in Ω that are **not it** A.



Mynd: Complement.

Probability

Probability

The **probability** of a certain outcome of a certain outcome of a random phenomena is the proportion of the cases when that the random phenomena gets that outcome when the phenomena is repeated often enough. This ratio can be at minimum **zero** and at maximum **one**.

Probability of an event

The **probability of an event** A, denoted $P(A)$, is the probability that the observed outcome will be in A.

Equally likely outcomes

Equally likely outcomes

Equally likely outcomes are only defined for random phenomena with **finite** Ω. Then the probability of every outcome in Ω is the same.

Probability of events when all outcomes are equally likely

If all of the outcomes of a random phenomena are **equally likely**, then the probability of an event A are:

$$P(A) = \frac{\text{number of outcomes in } A}{\text{number of outcomes in } \Omega}$$

Formulas

Formulas

1. $P(\Omega) = 1$
2. $P(A^C) = 1 - P(A)$
3. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
4. If A and B are disjoint, $P(A \cup B) = P(A) + P(B)$

Conditional probability

Conditional probability

With $P(A|B)$ we denote the probability that event A occurs, given that event B has occurred. The probability of $P(A|B)$ can be calculated with

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \quad \text{if } P(B) > 0.$$

Probability of intersection of events

$$P(A \cap B) = P(A|B)P(B), \quad \text{if } P(B) > 0.$$

Independent events

Independent events

We say that events A and B are **independent** if the probability that an event A occurs does not change even though the event B has occurred and vice versa.

Probability of independent events

If A and B are independent, then

$$P(A \cap B) = P(A) \cdot P(B)$$

Random variable

Random variable

Random variable describes the outcome of a variable before it is measured

Syntax for random variables

Random variables are denoted with **capital** letters, often X
Values that a random variable **has received** are denoted with **lower-case** letters, often x
The same letter is always used for a random variable and the value it has received.

Discrete and continuous random variables

Discrete random variables

Discrete random variables describe discrete variables. They have a finite set of possible outcomes on every limited interval.

Continuous random variables

Continuous random variables describe continuous variables. They can obtain any outcome on some interval.

Syntax for the probability of random variables

Syntax for the probability of random variables

$P(X \leq a)$: Denotes the probability that the outcome of a random variable X will be **less or equal** then the value a.

$P(X \geq a)$: Denotes the probability that the outcome of a random variable X will be **greater or equal** then the value a.

$P(a \leq X \leq b)$: Denotes the probability that the outcome of a random variable X will be **between** $a$and b, both values included

$P(X = a)$: Denotes the probability that the outcome of a random variable X will be **exactly** the value a.

Probability distribution of random variables

Probability distribution random variables

The **Probability distribution** of a random variable is a rule that tells us which values a random variable can receive and furthermore:

$P(X = a)$ for all values a it can receive if the probability distribution is **discrete**.

$P(a \leq X \leq b)$ for all values a and b if the probability distribution is **continuous**.

The probability distribution of a random variable gives us all available information possible of the random variable!
Why do you think that we define the probability distribution in a different manner depending on whether the random variable is discrete or continuous?

Types of probability distributions

Types of probability distributions

The randomness of many of the variables that we investigate are similar by nature.
Then the random variables that they describe behave similarly.
As a consequence, they will have similar probability distributions.
Then we say that the probability distributions of the random variables are of the same type.

Parameter

Parameter

Every type of probability distribution is described with numbers that are called the **parameters** of the probability distribution.
Different parameters describe different probability distributions .
Normally the parameters are only one or two.
If we know the type of the probability distribution of a random variable, the values of its parameters give all information available about the random variables.

Short summary

- One can talk about the probability that a random variable receives certain values.
- That probability is described by the probability distribution of the random variables, that give all information available about the random variables.
- Many random variables have probability distributions of a known type.
- Every type of probability distribution is described with numbers that are called parameters.
- To every type of probability distributions belong certain parameters and they are normally one or two.
- If we know the type of the probability distribution of a random variable, the values of its parameters give all information available about the random variables.

Independent random variables

Independent random variables

We say that two random variables are **independent** if the outcome of one random variable does not affect the outcome of the other random variable.

Dependent random variables

We say that two random variables are **dependent** if they are not independent, that is, if the outcome of one random variable does not affect the outcome of the other random variable or vice versa.

Independent and identically distributed random variables

We say that random variables $X_1, \ldots, X_n$ are **independent** if each of them is independent to all of the others and **identically distributed** if they all have the same probability distribution.

Expected value of a random variable

Expected value of a random variables

Expected value of a random variable is the **true** mean of the random variable. It is either denoted with μ or $E[X]$. It is also called **population mean** when appropriate.

Law of large numbers

As the number of measurements of a random variable X grows, the mean of the measurements, denoted $\bar{x}$, gets closer to the **expected value** of the random variable, denoted μ or $E[X]$.

Expected value of a discrete random variable

Expected value of a discrete random variable

If a random variable is **discrete** its expected value is the weighted mean of all of its possible outcomes, where the weight of each outcome is the probability that the random variable receives that outcome.

Formula for the expected value of a discrete random variable

If a random variable X is discrete, then its expected value is

$$\mu = \sum x_i \cdot P(X = x_i)$$

where we sum over all possible outcomes of the random variable.

Variance of random variables

Variance of random variables, $Var[X]$

As random variables have true means, they also have a **true variance**. It is either denoted with σ^2, or $Var[X]$. It is also called the **population variance** when appropriate.

Formula for the variance of a discrete random variable

The **variance** of a discrete random variable is

$$\sigma^2 = \sum (x_i - \mu)^2 \cdot P(X = x_i)$$

where we sum over all possible outcomes of the random variable.