Discrete probability distributions

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Main topics:

1. Mass function
2. Formulas for discrete random variables
3. Bernoulli trial and the binomial distribution
4. The Poisson distribution

Probability distributions of random variables

- Many random variables have probability distributions of a known type.
- The probability distributions of random variables are discrete if the random variables are discrete and continuous if not.
- Let us look at the two most common discrete probability distributions:
  - **The binomial distribution**
  - **The Poisson distribution**
- We will see how these two probability distributions can be used to describe several random phenomena.

Mass function

Mass function

Discrete probability distributions are described with a **mass function** and we will use it to calculate that probability of certain outcomes of discrete random variables. We denote the mass function with $f(x)$ and it can be written as

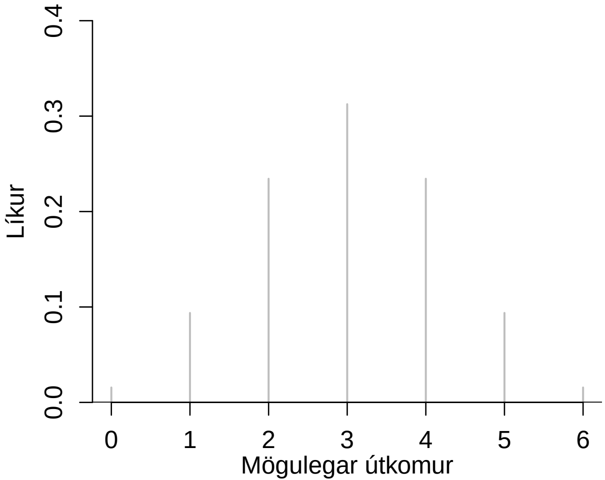
$$f(x) = P(X = x).$$

The following holds for the mass function:

$$f(x) \geq 0$$
$$\sum_{\text{yfir öll x}} f(x) = 1.$$

We use barplots to represent mass functions graphically.

Barplot of a mass function



Formulas for discrete random variables

Formulas fyrir discrete random variables

When calculating probabilities for a discrete random variable X calculations can often be simplified by "turning around" the probabilities

$$\begin{aligned}P(X \leq k) &= 1 - P(X > k)\\P(X < k) &= 1 - P(X \geq k)\\P(X \geq k) &= 1 - P(X < k)\\P(X > k) &= 1 - P(X \leq k)\end{aligned}$$

where k can be any number in the outcome space of X.

Bernoulli trial

Bernoulli trial

Every event in a group of repeated events is classified as a **Bernoulli trial** if the following holds:

1. Every event has only two possible outcomes. These outcomes are traditionally called **success** and **failure**. An event is successful if the outcome is a **success** and unsuccessful if its outcome is a **failure**.
2. The probability of a success are the same for every event. The probability of a failure is therefore the same for all events as the probability of a failure is always 1 minus the probability of a success.
3. An outcome in one event does not influence the outcome of another event, that is the events are independent.

The binomial distribution

- We are often interested in calculating how many successful events are among a set of Bernoulli trials.
- We would for example want to calculate the probability of receiving two sixes (which would be the success) when a dice is thrown five times.
- We view the total number of successful events as a random variable X.
- It has a known probability distribution that is called the **binomial distribution** and it is described with the parameters n which is the total number of Bernoulli trials that are conducted, and p which is the probability that is the probability of success within the Bernoulli trials.

The binomial distribution

The binomial distribution

Let the random variable X denote the number of successful events from n Bernoulli trials. Then X follows a binomial distribution with the parameters n and p, written $X \sim B(n,p)$, where p is the probability of success within each event. The probability that the random variable X receives the value $k \in 0, 1, 2, ...n$ can be calculated with the mass function of the binomial distribution:

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}, \qquad k = 0, 1, 2, ...n$$

$\binom{n}{k}$ the binomial coefficient. It is the probability of receiving k positive outcomes in n events and calculated with

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Where $k! = k \cdot (k-1) \cdot (k-2) \cdot ... \cdot (1)$. Notice that $0! = 1$.

The binomial distribution

- We have now seen that the probability that the random variable X receives a certain value k can be calculated.
- In addition to calculating $P(X = k)$ we are often interested in calculating
  - $P(a \leq X \leq b)$ or $P(a < X < b)$
  - $P(X \leq k))$ or $P(X < k)$
  - $P(X \geq k))$ or $P(X > k)$
- We can calculate all of those probabilities by using the formula for the mass function of a binomial distribution along with the rules on slide 6.

Expected value and variance of a binomial distribution

Expected value and variance of a binomial distribution

If X follows a binomial distribution, $X \sim B(n, p)$ then

$$\mathsf{E}[X] = np$$
$$\mathsf{Var}[X] = np(1-p)$$

The Poisson distribution

- The Poisson distribution is often used to describe the number of random phenomena that occur within a **certain unit** but the number of possible outcomes has no upper limit.
- The units can be **time intervals**, **spatial intervals** or some **physical object**.
- As an example we can mention the number of phone calls an office receives every minute, the number of reindeers per each square kilometer or the number of typos on each page.

The Poisson distribution

The Poisson distribution

The Poisson distribution has one parameter that is called λ. If X follows a Poisson distribution with the parameter λ the probability that the random variable X receives a value k, $k = 0, 1, 2, ...$ with the mass function of the Poisson distribution:

$$P(X = k) = \frac{e^{-\lambda}\lambda^k}{k!}.$$

We write $X \sim \text{Pois}(\lambda)$. The sample space of X is $\Omega = \{0,1,2,...\}$.
The parameter λ is the expected value of the random variable X, that is, its true mean. It describe how many successful outcomes we expect on average per each unit.

The Poisson distribution

- We have now seen that the probability that a random variable X that follows the Poisson distribution receives a certain value k can be calculated with the mass function of the Poisson distribution.
- We are often interested in calculating other probabilities:
    - $P(a \leq X \leq b)$ or $P(a < X < b)$
    - $P(X \leq k)$ or $P(X < k)$
    - $P(X \geq k)$ or $P(X > k)$.

We can calculate all of these probabilities with the mass function of the Poisson distribution.

Changing units

- When calculating the probability that a random variable that follows a Poisson distribution receives a certain value, we often have given the value of λ in another unit then the one we wish to use.
- We could for example know that the number of car incidents in Reykjavik every week, but we wish to know the number of incidents per day. Then λ need to be adjusted to a new unit.
- If the new unit is a times the old unit, then

$$\lambda_{\text{new}} = a \cdot \lambda_{\text{old}}$$

Where λ_{old} is the "old λ" and λ_{new} is the "new λ" adjusted to a new unit.

The expected value and variance of a Poisson distribution

Expected value and variance of a Poisson distribution

If X follows a Poisson distribution, $X \in \text{Pois}(\lambda)$ then

$$\begin{aligned} \text{E}[X] &= \lambda \\ \text{Var}[X] &= \lambda. \end{aligned}$$