Inference regarding means and variances

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Main topics:

1. Inference regarding variances
2. Inference regarding mean of a population
3. Inference regarding means of two populations
4. Independent measurements
5. Paired measurements

Introduction

- We will start to discuss inference on the variance of a normally distributed population and how to compare the variances in two normally distributed populations.
- First we will discuss confidence intervals and hypothesis test for the variance of a normally distributed population.
- Then we explore hypothesis tests that can be used when comparing the variances of two normally distributed populations.

Conducting hypothesis tests

Conducting hypothesis tests

1 Decide which hypothesis test is appropriate for our measurements.

2 Decide the α-level.

3 Propose a null hypothesis and decide the direction of the test (one- or two-sided).

4 Calculate the test statistic for the hypothesis test.

5a See whether the test statistic falls within the rejection interval.

5b Look at the p-value of the test statistic.

6 Draw conclusions.

Inference on the variance of a population

- In this section we discuss hypothesis tests and confidence intervals that apply when making inference on the variance of a normally distributed population, σ^2.
- When calculating confidence intervals and testing hypothesis for the variance of a population, the χ^2-distribution is used.
- The null hypothesis in this section is that the variance of the population equals some specific value that we denote σ_0^2.
- The null hypothesis is written $H_0 : \sigma^2 = \sigma_0^2$.
- It depends on the direction of the hypothesis test what conclusion are drawn if the null hypothesis is rejected.
- If the hypothesis test is two-sided we conclude that the variance of the population, σ^2, differs from σ_0^2 but if it is one-sided we can only conclude that the variance is greater or less than σ_0 depending on the case.

Confidence interval for the variance of a population

Confidence interval for the variance of a population

The lower bound of a $1 - \alpha$ confidence interval is: pace-2mm

$$\frac{(n-1) \cdot s^2}{\chi^2_{1-\alpha/2,(n-1)}}$$

The upper limit of a $1 - \alpha$ confidence interval is: pace-2mm

$$\frac{(n-1) \cdot s^2}{\chi^2_{\alpha/2,(n-1)}}$$

The confidence interval can thus be written: pace-2mm

$$\frac{(n-1) \cdot s^2}{\chi^2_{1-\alpha/2,(n-1)}} \ < \ \sigma^2 \ < \ \frac{(n-1) \cdot s^2}{\chi^2_{\alpha/2,(n-1)}}$$

where n is the number of measurements in the sample and $s^2$is the sample variance. $\chi^2_{1-\alpha/2,(n-1)}$ and $\chi^2_{\alpha/2,(n-1)}$ is found in the χ^2-table.

Inference on the variance of a population

Inference on the variance of a population

The null hypothesis is:

$$H_0 : \sigma^2 = \sigma_0^2$$

The test statistic is:

$$\chi^2 = \frac{(n-1)S^2}{\sigma_0^2}$$

The alternative hypothesis and the rejection areas are:

Alternative hypothesis	Reject H_0 if:
$\mathrm{H}_1 : \sigma^2 < \sigma_0^2$	$\chi^2 < \chi^2_{\alpha,(n-1)}$
$\mathrm{H}_1 : \sigma^2 > \sigma_0^2$	$\chi^2 > \chi^2_{1-\alpha,(n-1)}$
$\mathrm{H}_1 : \sigma^2 \neq \sigma_0^2$	$\chi^2 < \chi^2_{\alpha/2,(n-1)}$ or $\chi^2 > \chi^2_{1-\alpha/2,(n-1)}$

Inference on the variance of two populations

- The hypothesis tests that we discuss in this section are used to compare the variance of two populations that both are normally distributed.
- Tests of this kind are often conducted before hypothesis tests where the means of two populations are compared and the variance of the populations is unknown and the samples are not large.
- The null hypothesis i this section is that the variance of the two populations is equal, written $H_0 : \sigma_1^2 = \sigma_2^2$.
- If the hypothesis test is two-sided we can draw the conclusion that the variances are unequal, but if it is one-sided we can only draw the conclusion that the variance in one sample is greater then the variance in the other sample.

Hypothesis tests for the variances of two populations

Hypothesis tests for the variances of two populations

The null hypothesis is:

$$H_0 : \sigma_1^2 = \sigma_2^2$$

The alternative hypothesis can be one-sided or two-sided and the test-statistic differs by what the alternative hypothesis is. Possible alternative hypothesis, test statistics and their rejection areas are shown below.

Alternative hypothesis	Test statistic	Reject H_0 if:
$\mathrm{H}_1 : \sigma_1^2 < \sigma_2^2$	$F = \frac{S_2^2}{S_1^2}$	$F > F_{1-\alpha,(n_2-1,n_1-1)}$
$\mathrm{H}_1 : \sigma_1^2 > \sigma_2^2$	$F = \frac{S_1^2}{S_2^2}$	$F > F_{1-\alpha,(n_1-1,n_2-1)}$
$\mathrm{H}_1 : \sigma_1^2 \neq \sigma_2^2$	$F = \frac{S_M^2}{S_m^2}$	$F > F_{1-\alpha/2,(n_M-1,n_m-1)}$

In the two-sided test, one shall denote the population with greater sample variance with M and the one with smaller sample variance with m.

Hypothesis tests for μ

- Now we will discuss hypothesis tests and confidence intervals that apply when making inference on the mean of a population, μ.
- We will use them for example to test the hypothesis that mean precipitation in Reykjavik in june is less then 50 mm, that the average heart rate of men over fifty years old is greater then 99 beats p/minute, that the average number of nights slept in hotels and hostels in june differs from 100000 and so and so forth.
- All hypothesis test that will be discussed have the same null hypothesis, that the mean of the population is equal to a certain value that is called μ_0.

Hypothesis tests for μ

The null hypothesis is written:

$$H_0 : \mu = \mu_0$$

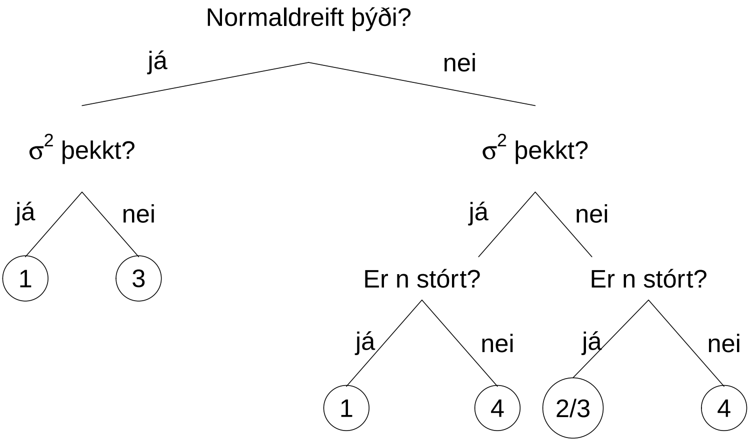
.

- It depends on the direction of the hypothesis test, what conclusions are made if we reject the null hypothesis.
- If the hypothesis test is two sided, we can conclude that the mean of the population, μ, differs from μ_0.
- If it is one sided we can only conclude that it is greater in one case or less in the other case then μ_0 depending on the case.

Hypothesis tests for μ

- Circumstances can be very different when we make inference on the mean of a population and we categorize them into four different cases, but each case is treated differently.
- The decision tree on slide 31 shows which case corresponds to which circumstance , but in order to select the appropriate case we need to answer three questions that are shown on slide **??**.
- The question are about the probability distribution of the population, whether it's variance, σ^2 is known and the size of the population, n. Each case is discussed separately in the following slides

Decision tree



Mynd: Decision tree for μ

Decision tree

These three questions are answered in correct order and the answers decide how we trace us down the decision tree.

1. Is the population normally distributed
   This need to be based on prior experience or by looking at the distribution of the sample and conclude from that. It can though be douptful if the sample is small.
2. Is the variance of the **population**, σ^2, known?
   Notice that this is rarely the case, although it may happen that such detailed prior investigations have been made that we can assume that the variance is known.
3. Is the sample large?
   We use the rule of thumb that n is large if $n > 30$. This is not a universal rule though.

Case 1

Case 1 corresponds to:

- When one can assume that the population follows a normal distribution and the variance (σ^2) of the distribution is known.
- when n is large and σ^2 is known, although the population is not normally distributed.

Confidence interval for μ - case 1

Confidence interval for μ - case 1

Lower bound of $1-\alpha$ confidence interval is:

$$\bar{x} - z_{1-\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}.$$

Upper bound of $1-\alpha$ confidence interval is:

$$\bar{x} + z_{1-\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}.$$

The confidence interval can thus be written as:

$$\bar{x} - z_{1-\alpha/2} \cdot \frac{\sigma}{\sqrt{n}} < \mu < \bar{x} + z_{1-\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}$$

where $\bar{x}$ is the sample mean and σ is the standard deviation of the population. $z_{1-\alpha/2}$ value is found in the standard normal distribution table.

Hypothesis test for μ - case 1

Hypothesis test for μ - case 1

The null hypothesis is:

$$H_0 : \mu = \mu_0$$

The test statistic is:

$$Z = \frac{\overline{X} - \mu_0}{\sigma/\sqrt{n}}$$

If the null hypothesis is true, the test statistic followsthe standardized normal distribution, or $Z \sim N(0,1)$.
The rejection areas are:

Alternative hypothesis	Reject H_0 if:
$\mathrm{H}_1 : \mu < \mu_0$	$Z < -z_{1-\alpha}$
$\mathrm{H}_1 : \mu > \mu_0$	$Z > z_{1-\alpha}$
$\mathrm{H}_1 : \mu \neq \mu_0$	$Z < -z_{1-\alpha/2}$ or $Z > z_{1-\alpha/2}$

μ - case 2

Case 2 corresponds to:

- when the sample is large and we do not know the variance of the population. We do not need to assume that the population is normally distributed.

Be careful! One can always calculate the variance of the **sample** but the variance of the **population** is rarely known!

μ - case 2

As the variance of the population is not known, we use the variance of the sample to estimate the variance of the population with

$$s^2 = \frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n-1}.$$

In order to find the standard deviation of the sample, we take the square root of the variance

$$s = \sqrt{s^2}.$$

Confidence interval for μ - case 2

Confidence interval for μ - case 2

Lower bound of $1-\alpha$ confidence interval:

$$\bar{x} - z_{1-\alpha/2} \cdot \frac{s}{\sqrt{n}}.$$

Upper bound of $1-\alpha$ confidence interval:

$$\bar{x} + z_{1-\alpha/2} \cdot \frac{s}{\sqrt{n}}.$$

The confidence interval can thus be written as:

$$\bar{x} - z_{1-\alpha/2} \cdot \frac{s}{\sqrt{n}} < \mu < \bar{x} + z_{1-\alpha/2} \cdot \frac{s}{\sqrt{n}}$$

where $\bar{x}$ og s are the sample mean and standard deviation. $z_{1-\alpha/2}$ value is found in the standard normal distribution table.

Hypothesis test for μ - case 2

Hypothesis test for μ - case 2

The null hypothesis is:

$$H_0 : \mu = \mu_0$$

The test statistic is:

$$Z = \frac{\overline{X} - \mu_0}{S/\sqrt{n}}$$

If the null hypothesis is true, the test statistic follows the standardized normal distribution, or $Z \sim N(0,1)$.
The rejection areas are:

Alternative hypothesis	Reject H_0 if:
$H_1 : \mu < \mu_0$	$Z < -z_{1-\alpha}$
$H_1 : \mu > \mu_0$	$Z > z_{1-\alpha}$
$H_1 : \mu \neq \mu_0$	$Z < -z_{1-\alpha/2}$ or $Z > z_{1-\alpha/2}$

μ - case 3

Case 3 corresponds to two cases:
In both cases, the variance (σ^2) of the **population**, to which the sample belongs, unknown. On the other hand, we either need to assume that:

- the population is normally distributed
- or that we have many measurements in our sample (then the population does not have to be normally distributed). This is the same as case 2.

When calculating confidence intervals and conducting hypothesis test in this case, one uses the t-distribution.

Of the overlap of case 2 and 3

- Notice that when case 2 (which uses z-test) is valid, one can successfully use case 3 instead (which uses t-test). This is because when the number of degrees of freedom is large, the t-distribution is similar to the normal distribution.
- T-test, unlike z-tests, are built in most statistical software and therefore more used.
- If we are doing calculations by hand it is often better to use z-tests because then we can easily calculate p-values.

Confidence interval for μ - case 3

Confidence interval for μ - case 3

Lower bound of $1-\alpha$ confidence interval:

$$\bar{x} - t_{1-\alpha/2,(n-1)} \cdot \frac{s}{\sqrt{n}}.$$

Upper bound of $1-\alpha$ confidence interval:

$$\bar{x} + t_{1-\alpha/2,(n-1)} \cdot \frac{s}{\sqrt{n}}.$$

The confidence interval can thus be written as:

$$\bar{x} - t_{1\alpha/2,(n-1)} \cdot \frac{s}{\sqrt{n}} < \mu < \bar{x} + t_{1-\alpha/2,(n-1)} \cdot \frac{s}{\sqrt{n}}$$

where $\bar{x}$ og s are the mean and the standard deviation of the sample and $t_{1-\alpha/2,(n-1)}$ value is found in the t-table.

Hypothesis test for μ - case 3

Hypothesis test for μ - case 3

The null hypothesis is:

$$H_0 : \mu = \mu_0$$

The test statistic is:

$$T = \frac{\overline{X} - \mu_0}{S/\sqrt{n}}$$

If the null hypothesis is true, the test statistic follows t-distribution with $(n-1)$ degree of freedom, or $T \sim t_{(n-1)}$.
The rejection areas are:

Alternative hypothesis	Reject H_0 if:
$\text{H}_1 : \mu < \mu_0$	$T < -t_{1-\alpha,(n-1)}$
$\text{H}_1 : \mu > \mu_0$	$T > t_{1-\alpha,(n-1)}$
$\text{H}_1 : \mu \neq \mu_0$	$T < -t_{1-\alpha/2,(n-1)}$ or $T > t_{1-\alpha/2,(n-1)}$

μ - case 4

In case 4 one can neither use z-test nor t-test unless further approximations are used. In these cases one can do one of the following:

- Transform the data
- Use nonparametric tests
- Check whether the population follows some other known distribution and use tests that are applicable for them.

Inference on the mean of two populations

- Now we will discuss hypothesis tests and confidence intervals that apply when making inference on the mean of two populations.
- The means are named μ_1 and μ_2 and we wish to make inference on their difference, $\mu_1 - \mu_2$.
- The tests applied can broadly be divided into two groups:
    - Tests for independent measurements.
    - Tests for paired measurements.

Independent or paired?

The first question we need to ask is if the measurements are independent or paired.

Examples of tests for independent measurements:

- Height of 50 men and 50 women used to test the hypothesis that men are on average taller then women.
- The heart rate of 30 women in the age 41-50 and 30 women in the age 51-60 measured to test the hypothesis that there is a difference in the heart rate of women in these two age groups.

Examples of tests for paired measurements:

- The weight of 30 men before they undergo an intensive workout program. The weight is measure again after the program to test the hypothesis that the workout is successful for loosing weight.
- The age of 40 men and their wives is noted to test the hypothesis that in marriages of men and women are the men on average older then the women.

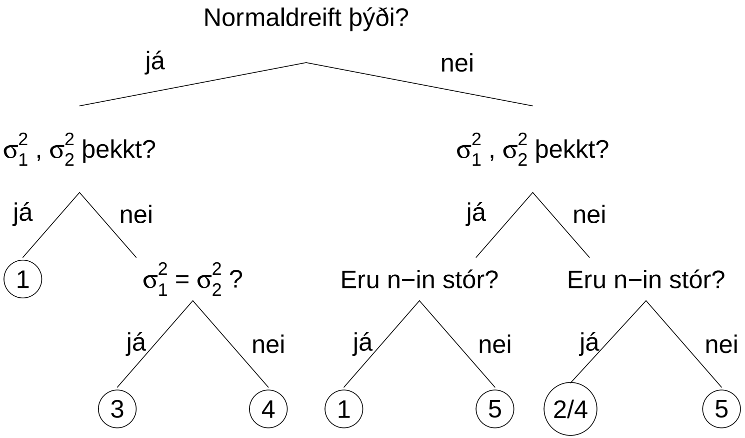
Independent measurements

- All hypothesis tests in this lecture test the same null hypothesis, whether the difference of the two means is equal to a certain value that we call δ.
- The null hypothesis is $H_0 : \mu_1 - \mu_2 = \delta$.
- It depends on the direction of the hypothesis test, what conclusions are made if we reject the null hypothesis.
- If the hypothesis test is two sided, we can conclude that the difference of the means, $\mu_1 - \mu_2$, differs from δ.
- If it is one sided we can only conclude that the difference is greater in one case or less in the other case then δ depending on the case.

Independent measurements

- As with one mean, we use different tests for different circumstances.
- The circumstances are categorized into five cases:
- The decision tree on slide 31 shows which case corresponds to which circumstance , but in order to select the appropriate case we need to answer four questions that are shown on slide **??**.
- We note the mean, variance and sample size of one population with μ_1, σ_1^2 and n_1 but the other with μ_2, σ_2^2 and n_2.

Decision tree - independent measurements



Mynd: Decision tree for $\mu_1 - \mu_2$

Decision tree - independent measurements

1. Are the populations normally distributed?
   This need to be based on prior experience or by looking at the distributions of the samples and conclude from that. It can though be douptful if the samples are small.
2. Is the variance of the **populations**, σ_1^2, σ_2^2, known?
   Notice that this is rarely the case, although it may happen that such detailed prior investigations have been made that we can assume that the variance is known.
3. Are the samples large?
   We use the rule of thumb that the samples are large if $n_1 > 30$ and $n_2 > 30$. This is not a universal rule though.
4. When the variances are unknown, can we yet assume that they are equal for the two populations, that is if $\sigma_1^2 = \sigma_2^2$?
   Later on we will see how to test this hypothesis formally, but until then we will use the rule of thumb that if one sample variance is more then four times greater then the other, we cannot assume that the variances of the populations are equal.

$\mu_1 - \mu_2$ - case 1

Case one applies when:

- it can be assumed that the populations are normally distributed and the variances of the populations, (σ_1^2 and σ_2^2) are known.
- when n_1 and n_2 are large and σ_1^2 and σ_2^2 are known, although the populations are not normally distributed.

Confidence interval for the difference of two means - case 1

Confidence interval for the difference of two means - case 1

Lower bound of $1-\alpha$ confidence interval is:

$$\bar{x}_1 - \bar{x}_2 - z_{1-\alpha/2} \cdot \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$$

Upper bound of $1-\alpha$ confidence interval is:

$$\bar{x}_1 - \bar{x}_2 + z_{1-\alpha/2} \cdot \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$$

Confidence interval for the difference of two means - case 1

Confidence interval for the difference of two means - case 1

The confidence interval is:

$$\bar{x}_1 - \bar{x}_2 - z_{1-\alpha/2} \cdot \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} < \mu_1 - \mu_2 < \bar{x}_1 - \bar{x}_2 + z_{1-\alpha/2} \cdot \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$$

where $\bar{x}_1$, $\bar{x}_2$ are the sample means and σ_1^2, σ_2^2 are the population variances. $z_{1-\alpha/2}$ is found in the standardized normal distribution table.

Hypothesis test for the difference of two means - case 1

Hypothesis test for the difference of two means - case 1

The null hypothesis is:

$$H_0 : \mu_1 - \mu_2 = \delta$$

The test statistic is:

$$Z = \frac{\overline{X}_1 - \overline{X}_2 - \delta}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$$

If the null hypothesis is true, the test statistic follows the standardized normal distribution, or $Z \sim N(0, 1)$.

Alternative hypothesis	Reject H_0 if:
$\mathrm{H}_1 : \mu_1 - \mu_2 < \delta$	$Z < -z_{1-\alpha}$
$\mathrm{H}_1 : \mu_1 - \mu_2 > \delta$	$Z > z_{1-\alpha}$
$\mathrm{H}_1 : \mu_1 - \mu_2 \neq \delta$	$Z < -z_{1-\alpha/2}$ or $Z > z_{1-\alpha/2}$

Notice that δ can be any number at all, but in most cases $\delta = 0$.

$\mu_1 - \mu_2$ - case 2

Case 2 applies when:

- we do not know the population variances (σ_1^2 and σ_2^2) but the samples are large. We do not need to assume that the populations are normally distributed.
- In this case one can successfully use case 4, but that is built in most statistical software (such as R). When calculating in hands case 2 is easier though.

$\mu_1 - \mu_2$ - case 2

As the variance of the population is not known, we use the variance of the sample to estimate the variance of the population with

$$s^2 = \frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n-1}.$$

In order to find the standard deviation of the sample, we take the square root of the variance

$$s = \sqrt{s^2}.$$

These values are calculated for each sample separately and named s_1 and s_2 as appropriate.

Confidence interval for the difference of the mean of two populations - case 2

Confidence interval for the difference of the mean of two populations - case 2

Lower bound of $1-\alpha$ confidence interval is:

$$\bar{x}_1 - \bar{x}_2 - z_{1-\alpha/2} \cdot \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$$

Upper bound of $1-\alpha$ confidence interval is:

$$\bar{x}_1 - \bar{x}_2 + z_{1-\alpha/2} \cdot \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$$

Confidence interval for the difference of the mean of two populations - case 2

Confidence interval for the difference of the mean of two populations - case 2

The confidence interval is:

$$\bar{x}_1 - \bar{x}_2 - z_{1-\alpha/2} \cdot \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}} < \mu_1 - \mu_2 < \bar{x}_1 - \bar{x}_2 + z_{1-\alpha/2} \cdot \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$$

where $\bar{x}_1$, $\bar{x}_2$ are the sample means and s_1^2, s_2^2 are the sample variances. $z_{1-\alpha/2}$ is found in the standardized normal distribution table.

Hypothesis test for the difference of the means of two populations - case 2

Hypothesis test for the difference of the means of two populations - case 2

The null hypothesis is:

$$H_0 : \mu_1 - \mu_2 = \delta$$

The test statistic is:

$$Z = \frac{\overline{X}_1 - \overline{X}_2 - \delta}{\sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}}}$$

If the null hypothesis is true, the test statistic follows the standardized normal distribution, or $Z \sim N(0,1)$.

Alternative hypothesis	Reject H_0 if:
$\text{H}_1 : \mu_1 - \mu_2 < \delta$	$Z < -z_{1-\alpha}$
$\text{H}_1 : \mu_1 - \mu_2 > \delta$	$Z > z_{1-\alpha}$
$\text{H}_1 : \mu_1 - \mu_2 \neq \delta$	$Z < -z_{1-\alpha/2}$ or $Z > z_{1-\alpha/2}$

$\mu_1 - \mu_2$ - case 3

Case 3 applies when:

- One can assume that the populations are normally distributed, the variances (σ_1^2 and σ_2^2) of the populations are unknown, but we assume that $\sigma_1^2 = \sigma_2^2$.

Later on we will see how to test this hypothesis formally, but until then we will use the rule of thumb that if one sample variance is more then four times greater then the other, we cannot assume that the variances of the populations are equal.
The t-distribution is used for calculating confidence intervals and hypothesis testing in this case.

$\mu_1 - \mu_2$ - case 3

Before we can calculate confidence intervals and conduct hypothesis tests we need to calculate the **pooled variance** of the samples, which is denoted s_p^2.

$$s_p^2 = \frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}.$$

where s_1^2 and s_2^2 are calculated in the same way as earlier.

Confidence interval for the difference of the mean of two populations - case 3

Confidence interval for the difference of the mean of two populations - case 3

Lower bound of $1-\alpha$ confidence interval is:

$$\bar{x}_1 - \bar{x}_2 - t_{1-\alpha/2,(n_1+n_2-2)} \cdot s_p\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}$$

Upper bound of $1-\alpha$ confidence interval is:

$$\bar{x}_1 - \bar{x}_2 + t_{1-\alpha/2,(n_1+n_2-2)} \cdot s_p\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}$$

where $\bar{x}_1$, $\bar{x}_2$ are the sample means and s_1^2, s_2^2 are the sample variances . $t_{1-\alpha/2,(n_1+n_2-2)}$ is found in the t-distribution table.

Hypothesis test for the difference of the means of two populations - case 3

Hypothesis test for the difference of the means of two populations - case 3

The null hypothesis is:

$$H_0 : \mu_1 - \mu_2 = \delta$$

The test statistic is:

$$T = \frac{\overline{X}_1 - \overline{X}_2 - \delta}{s_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

If the null hypothesis is true, $T \sim t_{(n_1+n_2-2)}$.

Alternative hypothesis	Reject H_0 if:
$\mathrm{H}_1 : \mu_1 - \mu_2 < \delta$	$T < -t_{1-\alpha,(n_1+n_2-2)}$
$\mathrm{H}_1 : \mu_1 - \mu_2 > \delta$	$T > t_{1-\alpha,(n_1+n_2-2)}$
$\mathrm{H}_1 : \mu_1 - \mu_2 \neq \delta$	$T < -t_{1-\alpha/2,(n_1+n_2-2)}$ or $T > t_{1-\alpha/2,(n_1+n_2-2)}$

$\mu_1 - \mu_2$ - case 4

Case 4 applies when:

- it can be assumed that the populations are normally distributed, the variances (σ_1^2 and σ_2^2) are unknown and we cannot assume that the variances are equal, or $\sigma_1^2 \neq \sigma_2^2$.
- when the variances (σ_1^2 and σ_2^2) are unknown but the samples are large. Then we don't have to assume that the samples are normally distributed. Then one can also use case 2, which is normally used when calculating by hands, but case 4 is used in most statistical software.

Later on we will see how to test this hypothesis formally, but until then we will use the rule of thumb that if one sample variance is more then four times greater then the other, we cannot assume that the variances of the populations are equal.

$\mu_1 - \mu_2$ - case 4

In this case the confidence interval and the test statistic resembles the one in case 2 but here it follows the t-distribution. The number of degrees of freedom in this t-distribution is denoted with ν and calculated by

$$\nu = \frac{\left(\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}\right)^2}{\frac{(s_1^2/n_1)^2}{n_1-1} + \frac{(s_2^2/n_2)^2}{n_2-1}}$$

where s_1^2 and s_2^2 are calculated by same methods as earlier. This hypothesis test is rarely done by hands but a statistical software used for the calculations.

Confidence interval for the difference of the mean of two populations - case 4

Confidence interval for the difference of the mean of two populations - case 4

Lower bound of $1-\alpha$ confidence interval is:

$$\bar{x}_1 - \bar{x}_2 - t_{1-\alpha/2,(\nu)} \cdot \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$$

Upper bound of $1-\alpha$ confidence interval is:

$$\bar{x}_1 - \bar{x}_2 + t_{1-\alpha/2,(\nu)} \cdot \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$$

Confidence interval for the difference of the mean of two populations - case 4

Confidence interval for the difference of the mean of two populations - case 4

The confidence interval is:

$$\bar{x}_1 - \bar{x}_2 - t_{1-\alpha/2,(\nu)} \cdot \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}} < \mu_1 - \mu_2 < \bar{x}_1 - \bar{x}_2 + t_{1-\alpha/2,(\nu)} \cdot \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$$

where $\bar{x}_1$, $\bar{x}_2$ are the sample means and s_1^2, s_2^2 are the sample variances . $t_{1-\alpha/2,(\nu)}$ is found in the t-table. ν is the number of degrees of freedom.

Hypothesis test for the difference of the means of two populations - case 4

Hypothesis test for the difference of the means of two populations - case 4

The null hypothesis is:

$$H_0 : \mu_1 - \mu_2 = \delta$$

The test statistic is:

$$T = \frac{\overline{X}_1 - \overline{X}_2 - \delta}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}$$

If the null hypothesis is true, the test statistic t-dreifingu með ν frígráðum or $T \sim t(\nu)$ where ν is calculated as shown in slide ??

Alternative hypothesis	Reject H_0 if:
$\mathrm{H}_1 : \mu_1 - \mu_2 < \delta$	$T < -t_{1-\alpha,(\nu)}$
$\mathrm{H}_1 : \mu_1 - \mu_2 > \delta$	$T > t_{1-\alpha,(\nu)}$
$\mathrm{H}_1 : \mu_1 - \mu_2 \neq \delta$	$T < -t_{1-\alpha/2,(\nu)}$ or $T > t_{1-\alpha/2,(\nu)}$

$\mu_1 - \mu_2$ - case 5

In case 5 one can neither use z-test nor t-test unless further approximations are made. In these cases one of the following can be done:

- Transform the data
- Use resampling methods
- Use nonparametric tests
- Test whether the measurements follow any known distributions and look at tests that apply to them.

Paired measurements

- Many statistical investigations are done on paired random samples.
- Those researches are ofter done in that manner that measurements are taken before and after some intervention. The hypothesis normally have the purpose to see whether the intervention was successful.
- We use paired tests to test these hypothesis.

Paired measurements

- Assume that we have n pair of measurements (X_i, Y_i), $i = 1, 2, 3 ... n$.
- We need to find the differences of these pairs:

$$D_i = X_i - Y_i.$$

  D_i is a random variable of size n from a population with mean μ_D.
- The hypothesis tests make inference on μ_D.

Paired measurements

Before conducting hypothesis tests, the following statistics need to be calculated:

$$\overline{D} = \frac{\sum_{i=1}^{n} D_i}{n}$$

which is the mean of the differences and

$${S_D}^2 = \frac{\sum_{i=1}^{n}(D_i - \overline{D})^2}{n-1}$$

is the standard deviation of the differences.

Paired measurements

- We test the null hypothesis that the mean of the differences is equal to a certain value that is denoted $\mu_{D,0}$.
- The null hypothesis is $H_0 : \mu_D = \mu_{D,0}$.
- It depends on the direction of the hypothesis test, what conclusions are made if we reject the null hypothesis.
- If the hypothesis test is two sided, we can conclude that the difference of the means, μ_D, differs from $\mu_{D,0}$.
- If it is one sided we can only conclude that the difference is greater in one case or less in the other case then $\mu_{D,0}$ depending on the case.

Paired measurements

- It depends on how many pairs of measurements we have en whether it can be assumed that the difference of the measurements is normally distributed if we use a z-test or a t-test to make inference on μ_D.
- If n is large, which here denotes the number of pairs, we can always use the z-test.
- The t-test can be used if μ_D is normally distributed and/or the sample is large.
- When we use a statistical software the t-test is preferred to the z-test when both tests are valid (when n is large).

Inference on paired measurements, n large

Inference on paired measurements, n large

The null hypothesis is:

$$H_0 : \mu_D = \mu_{D,0}$$

The test statistic is:

$$Z = \frac{\overline{D} - \mu_{D,0}}{S_D/\sqrt{n}}$$

If the null hypothesis is true, the test statistic follows the standardized normal distribution, or $Z \sim N(0,1)$.

Alternative hypothesis	Reject H_0 if:
$\mathrm{H}_1 : \mu_D < \mu_{D,0}$	$Z < -z_{1-\alpha}$
$\mathrm{H}_1 : \mu_D > \mu_{D,0}$	$Z > z_{1-\alpha}$
$\mathrm{H}_1 : \mu_D \neq \mu_{D,0}$	$Z < -z_{1-\alpha/2}$ or $Z > z_{1-\alpha/2}$

$z_{1-\alpha/2}$ is found in the standardized normal distribution table.

Inference on paired measurements, normally distributed differences and/or large n

When n is small the difference of the measurements need to be normally distributed.

Inference on paired measurements, normally distributed differences and/or large n

The null hypothesis is:

$$H_0 : \mu_D = \mu_{D,0}$$

The test statistic is:

$$T = \frac{\overline{D} - \mu_{D,0}}{S_D/\sqrt{n}}$$

If the null hypothesis is true, the test statistic is t-distributed with $(n-1)$ degrees of freedom, or $T \sim t_{(n-1)}$.

Inference on paired measurements, normally distributed differences and/or large n

Inference on paired measurements, normally distributed differences and/or large n

Alternative hypothesis	Reject H_0 if:
$\text{H}_1 : \mu_D < \mu_{D,0}$	$T < -t_{1-\alpha,(n-1)}$
$\text{H}_1 : \mu_D > \mu_{D,0}$	$T > t_{1-\alpha,(n-1)}$
$\text{H}_1 : \mu_D \neq \mu_{D,0}$	$T < -t_{1-\alpha/2,(n-1)}$ or $T > t_{1-\alpha/2,(n-1)}$

$t_{1-\alpha/2,(n-1)}$ is found in t-table