Problem statement and estimators

stats545.1 545.1 Point estimation and variances in the linear model

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Multiple linear regression problem

For y-observations, we want descriptive and predictive linear model of several variables

$y = \beta_1 x_1 + \beta_2 x_2 + \ldots + \beta_p x_p$

or, rather $y_i = \beta_1 x_{i1} + \beta_2 x_{i2} + \ldots + \beta_p x_{ip} + e_i$

Formulate with matrices...

$$\mathrm{y} = \mathrm{X}\boldsymbol{\beta} + \mathrm{e}$$

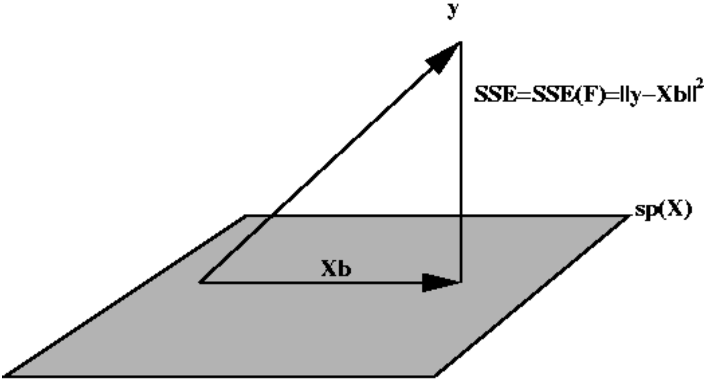
Note that intercept is implicit...

Statistical assumptions will be handled later!

Example: When a straight line is not an appropriate model for explaining the relationship between pairs of measurements, (x_i, y_i), it is possible to consider a quadratic response function, i.e. define the model $EY_i = \alpha + \beta x_i + \gamma x_i^2, \quad i = 1, \ldots, n$.

Example: Consider the data set (from Stefansson, Skuladottir and Petursson) of indices from Icelandic waters. Here T=temperature, U=catch per unit effort of (adult) shrimp, I=index of juvenile shrimp abundance, Y=catch of shrimp, B=biomass of capelin, G=measure of growth of cod from age 4 to 5, S=biomass of spawning cod, J=biomass of juvenile (immature) cod.

Geometric visualization of the multiple regression problem



Want β to give

$$min_{b_1,\ldots,b_p} \sum_{i=1}^{n} \left(y_i - (b_1 x_{i1} + b_2 x_{i2} + \ldots + b_p x_{ip})\right)^2 .$$

i.e. minimize

$$\min ||y - Xb||^2$$

Normal equations

Have

$$X'X\hat{\beta} = X'y$$

The solution

Solution:

$$\hat{\beta} = \left(X'X\right)^{-1} X'y$$

Prediction:

$$\hat{y} = X\hat{\beta} = X\left(X'X\right)^{-1} X'y.$$

Estimated residuals:

$$\hat{e} = y - \hat{y} = y - X\hat{\beta} = \left(I - X\left(X'X\right)^{-1} X^{T}\right) y.$$

... When the matrix is of full rank!

Sums of squares and norms

Sum of squared errors

$$SSE = ||\hat{e}||^2 = \sum_i (y_i - \hat{y}_i)^2 .$$

Denote SSE by $SSE(F)$ or $SSE(R)$ when comparing models.

Projection matrices

Projecton, "hat", matrix onto $V = sp(X)$:

$$H = X(X'X)^{-1}X'$$

and onto $V^{\perp} = sp(X)^{\perp}$:

$$I - H = I - X(X'X)^{-1}X'$$

References Neter, J., Kutner, M. H., Nachtsheim, C. J. and Wasserman, W. 1996. Applied linear statistical models. McGraw-Hill, Boston. 1408pp.