Estimable functions

stats545.1 Theory of linear models

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Estimable functions: The problem

If $\mathbf{X}$ is not of full rank, then the LS problem does not have a unique solution for $\hat{\boldsymbol{\beta}}$.

In general not all combinations of the form $\mathbf{c}'\hat{\boldsymbol{\beta}}$ may have unique solutions.

A linear combination $\mathbf{c}'\boldsymbol{\beta}$ is an **estimable function** if there is a vector of numbers, $\mathbf{a}$, such that

$$E[\mathbf{ay}] = \mathbf{c}'\boldsymbol{\beta}$$

for all $\boldsymbol{\beta}$.

NB: Viewed as a function of the unknown parameter vector, $\boldsymbol{\beta}$.
NB: The E-operator depends on $\boldsymbol{\beta}$, could write $g(\boldsymbol{\beta}) = \mathbf{c}'\boldsymbol{\beta}$ and require $g(\boldsymbol{\beta}) = E_{\boldsymbol{\beta}}[\mathbf{ay}] \quad \forall \boldsymbol{\beta}$ for some $\mathbf{a}$.

$$y_{ij} = \mu + \alpha_i + \epsilon_{ij}.$$

$$y_{ij} = \alpha + \beta x_{ij} + \beta_i x_{ij} + \epsilon_{ij}$$

Classification of estimable functions

Theorem: A parametric function $\psi = \mathbf{c}'\beta$ is estimable if and only if $\mathbf{c}' = \mathbf{a}'\mathbf{X}$ for some $\mathbf{a} \in \mathbf{R}^n$.

Gauss-Markov theorem

Theorem: (Gauss-Markov theorem): Let $EY = X\beta$, $VY = \sigma^2 I$. Then every estimable function $c'\beta$ has a unique unbiased linear estimate which has minimum variance in the class of all unbiased linear estimates. This estimate can be written the form $c'\hat{\beta}$ where $\hat{\beta}$ is any LS estimator.

Testing hypotheses in the linear model

$$\underbrace{\mathbf{y}}_{n\times 1} \sim n(\underbrace{\mathbf{X}}_{n\times p}\underbrace{\boldsymbol{\beta}}_{p\times 1}, \sigma^2 \underbrace{\mathbf{I}}_{n\times n})$$

Theorem: $\hat{\psi} \sim n\left(\psi, \boldsymbol{\Sigma}_{\hat{\psi}}\right)$, $\frac{||\mathbf{y}-\mathbf{X}\hat{\beta}||^2}{\sigma^2} \sim \chi^2_{n-r}$ and these two quantities are independent.

Confidence ellipsoids

References Neter, J., Kutner, M. H., Nachtsheim, C. J. and Wasserman, W. 1996. Applied linear statistical models. McGraw-Hill, Boston. 1408pp.
Scheffe, H. 1959. The analysis of variance. John Wiley and Sons, Inc, New York. 477pp.