Estimable functions

stats545.3 545.3 Hypothesis tests in the linear model, model building and predictions

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Estimable functions: The problem

If X is not of full rank, then the LS problem does not have a unique solution for $\hat{\boldsymbol{\beta}}$.

In general not all combinations of the form $\mathrm{c}'\hat{\boldsymbol{\beta}}$ may have unique solutions.

A linear combination $\mathrm{c}'\boldsymbol{\beta}$ is an **estimable function** if there is a vector of numbers, a, such that

$$E\left[\mathrm{a}'\mathrm{y}\right] = \mathrm{c}'\boldsymbol{\beta}$$

for all $\boldsymbol{\beta}$.

NB: Viewed as a function of the unknown parameter vector, $\boldsymbol{\beta}$.
NB: The E-operator depends on $\boldsymbol{\beta}$, could define $g(\boldsymbol{\beta}) := \mathrm{c}'\boldsymbol{\beta}$ and $h(\boldsymbol{\beta}) := E_{\boldsymbol{\beta}}[\mathrm{a}'\mathrm{y}]$ and require $g(\boldsymbol{\beta}) = h(\boldsymbol{\beta}) \quad \forall \boldsymbol{\beta}$ for some a.

$$y_{ij} = \mu + \alpha_i + \epsilon_{ij}.$$

Classification of estimable functions

Theorem: A parametric function $\psi = \mathrm{c}'\boldsymbol{\beta}$ is estimable if and only if $\mathrm{c}' = \mathrm{a}'\mathrm{X}$ for some $\mathrm{a} \in \mathrm{R}^n$.

Gauss-Markov theorem

Theorem: (Gauss-Markov theorem): Let $EY = X\beta$, $VY = \sigma^2 I$. Then every estimable function $c'\beta$ has a unique unbiased linear estimate which has minimum variance in the class of all unbiased linear estimates. This estimate can be written the form $c'\hat{\beta}$ where $\hat{\beta}$ is any LS estimator.

Testing hypotheses in the linear model

Theorem: If $\underbrace{\mathrm{y}}_{n\times 1} \sim n(\underbrace{\mathrm{X}}_{n\times p}\underbrace{\boldsymbol{\beta}}_{p\times 1}, \sigma^2\underbrace{\mathrm{I}}_{n\times n})$ and $\hat{\boldsymbol{\psi}}$ is a vector of estimable functions, then $\hat{\boldsymbol{\psi}} \sim n\left(\boldsymbol{\psi}, \Sigma_{\hat{\boldsymbol{\psi}}}\right)$, $\frac{||\mathrm{y}-\mathrm{X}\hat{\boldsymbol{\beta}}||^2}{\sigma^2} \sim \chi^2_{n-r}$ and these two quantities are independent.

References Neter, J., Kutner, M. H., Nachtsheim, C. J. and Wasserman, W. 1996. Applied linear statistical models. McGraw-Hill, Boston. 1408pp. Scheffe, H. 1959. The analysis of variance. John Wiley and Sons, Inc, New York. 477pp.