# Multi-factor ANOVA and interactions

## (STATS546.2: Applied analysis of variance (work in progress))

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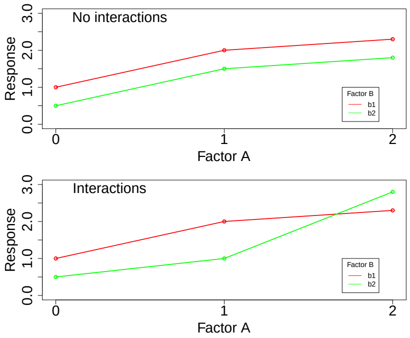
# What is an interaction?

In some cases there can be interactions between two or more factors.

Interactions occur when the effect of one factor is influenced by another.

When there are interaction between two factor the are said to be dependent.



# Example

# The two-way ANOVA model with interactions

ANOVA model with two factors and interactions is on the form

$$y_{ijk} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \epsilon_{ijk}$$

where

$$(\alpha\beta)_{ij} = \mu_{ij} - (\mu + \alpha_i + \beta_j)$$

# Fitting a model with interactions

The model is

$$y_{ijk} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \epsilon_{ijk}$$

The parameters are estimated as follows

$$\hat{\mu} = \bar{y}_{...}$$

$$\hat{\alpha}_i = \bar{y}_{i..} + \bar{y}_{...}$$

$$\hat{\beta}_j = \bar{y}_{.j.} + \bar{y}_{...}$$

$$(\hat{\alpha\beta})_{ij} = y_{\bar{y}_{ij.} - \bar{y}_i} - \bar{y}_{.j.} + \bar{y}_{...}$$

# The ANOVA table with interactions

| Source | df | SS |
|---|---|---|
| Factor A | $a-1$ | $SSA = nb\sum(\bar{y}_{i\cdot\cdot} - \bar{y}_{\cdots})^2$ |
| Factor B | $b-1$ | $SSB = na\sum(\bar{y}_{\cdot j\cdot} - \bar{y}_{\cdots})^2$ |
| Interactions AB | $(a-1)(b-1)$ | $SSAB = n\sum\sum(\bar{y}_{ij\cdot} - \bar{y}_{i\cdot\cdot} - \bar{y}_{\cdot j\cdot} + \bar{y}_{\cdots})$ |
| Error | $ab(n-1)$ | $SSE = \sum\sum\sum(y_{ijk} - \bar{y}_{ij\cdot})^2$ |
| Total | $n-1$ | $SSTOT = \sum\sum\sum(y_{ijk} - \bar{y}_{\cdots})^2$ |

# Hypothesis

When we have two factors there are three questions of interest.
Are there interactions between the two factors

$$H_0 : \text{All}(\alpha\beta)_{ij} = 0$$

$$H_1 : \text{Not all}(\alpha\beta)_{ij} = 0$$

Is there an effect of factor A

$$H_0 : \text{All}\alpha_i = 0$$

$$H_1 : \text{Not all}\alpha_i = 0$$

Is there an effect of factor B

$$H_0 : \text{All}\beta_j = 0$$

$$H_1 : \text{Not all}\beta_j = 0$$

# The F-test

To test for interaction and factor effects a F-test is applied. First it is tested if the interactions are significant.
$F = \frac{MSAB}{MSE}$ follows a f-distribution with $(a-1)(b-1)$ and $ab(n-1)$ degrees of freedom and if $F > F_{1-\alpha,(a-1)(b-1),ab(n-1)}$ the null hypothesis of no interaction is rejected.

If the null hypothesis of no interactions is rejected there is no need in tested the other two hypothesis as there is a effect of both factors because of the interactions.

If the null hypothesis is not rejected test for factors A and B are done as in two-way ANOVA without interactions.

Note that a test for interactions can only be carried out if there are two or more observation per cell.

# Example

# Assumptions

# Multiple comparisons when interactions are significant

When interactions are significant, factor levels means can not be compared as before.

Levels means for factor A need to be compared for each level of factor B individually.

Likewise, levels means for factor B need to be compared for each level of factor A individually.

# Tukey and Bonferroni

When interactions are significant and all pairwise comparisons are of interest the following procedure can be followed to calculate $1 - \alpha$ confidence limit of the difference.

$$\bar{y}_{ij}. - \bar{y}_k l. \pm \frac{1}{\sqrt(2)} q_{1-\alpha,ab,ab(n-1)} \sqrt{\frac{2MSE}{n}}$$

When only some c number of differences are of interest Bonferroni confidence intervals may be preferable

$$\bar{y}_{ij}. - \bar{y}_k l. \pm t_{1-\alpha/2c,ab(n-1)} \sqrt{\frac{2MSE}{n}}$$

# Example - Breaks in yarn

# Multi-factor ANOVA

There is really no limit how many factor can be included in ANOVA.

Interpretations become more difficult as number of factor increases.

ANOVA with 4 factors can have interactions between all of the factors which can be difficult to interpret.

Here we will only describe ANOVA with 3 factors.

# Example

# Model for three-way ANOVA

The model when we have three factors is on the form

$$y_{ijkl} = \mu + \alpha_i + \beta_j + \gamma_k + (\alpha\beta)_{ij} + (\alpha\gamma)_{ik} + (\beta\gamma)_{jk} + (\alpha\beta\gamma)_{ijk} + \epsilon_{ijkl}$$

# Estimating parameters

# Checking the assumptions

$: \text{All } (\alpha\beta\gamma)_{ijk} = 0$

$H_1 : \text{Not all } (\alpha\beta\gamma)_{ijk} = 0$

$H_0 : \text{All } (\alpha\beta)_{ij} = 0$ $H_0 : \text{All } (\alpha\gamma)_{ik} = 0$ $H_0 : \text{All } (\beta\gamma)_{jk} = 0$

$H_1 : \text{Not all } (\alpha\beta)_{ij} = 0$ $H_1 : \text{Not all } (\alpha\gamma)_{ik} = 0$ $H_1 : \text{Not all } (\beta\gamma)_{jk} = 0$

$H_0 : \text{All } \alpha_i = 0$ $H_0 : \text{All } \beta_j = 0$ $H_0 : \text{All } \gamma_k = 0$

$H_1 : \text{Not all } \alpha_i = 0$ $H_1 : \text{Not all } \beta_j = 0$ $H_1 : \text{Not all } \gamma_k = 0$

# ANOVA table for three-way ANOVA

ANOVA table when we have three factors become large and we use statistical software for the calculations.

| Source | df | SS |
|---|---|---|
| Factor A | $a-1$ | $SSA$ |
| Factor B | $b-1$ | $SSB$ |
| Factor C | $c-1$ | $SSC$ |
| Interactions AB | $(a-1)(b-1)$ | $SSAB$ |
| Interactions AC | $(a-1)(c-1)$ | $SSAC$ |
| Interactions BC | $(b-1)(c-1)$ | $SSBC$ |
| Interactions ABC | $(a-1)(b-1)(c-1)$ | $SSABC$ |
| Error | $abc(n-1)$ | $SSE$ |
| Total | $abcn-1$ | $SSTOT = \sum\sum\sum(y_{ijk} - \bar{y}\ldots)$ |

# Test the hypothesis

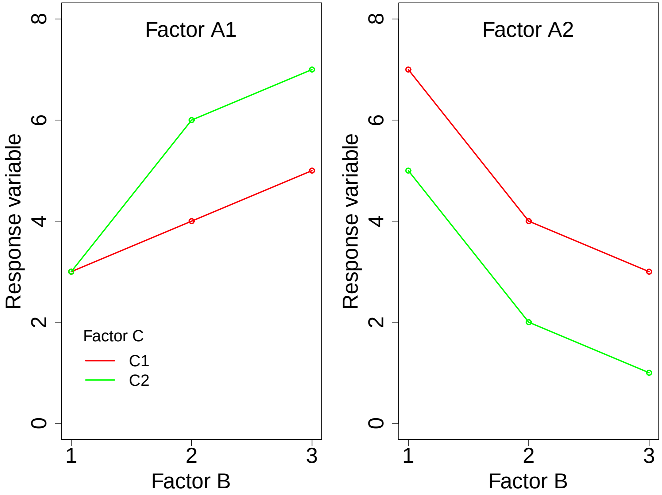
Use F-test to test hypothesis.

Start by testing if there is a 3-way interaction.

If 3-way interaction is significant than all factors have a significant effect and no further tests are required.

If not test the two way interactions etc...

# Interpretation of 3-way interactions



# Factors effect when 3-way interaction is significant

When 3-way interactions are significant the effect of the factors has to be analyzed jointly.

Difference between the $\mu_{ijk}$ has to be examined.

Bonferroni or Tukey can be used for multiple comparisons.

# Factors effect when 2-way interactions are significant

If there is one 2-way interaction significant then the factors involved has to be examined jointly.

If factors A and B interact $\mu_{ij.}$ has to be examined and the third factor can be examined individually.

If two or three 2-way interactions are significant all the factors need to be examined jointly.

It is then necessary to examine $\mu_{ijk}$.

# Factor effects without interactions.

When no interactions are significant each factor can be examined individually.

Each $\mu_{i..}$, $\mu_{.j.}$ and $\mu_{..k}$ can be examined.

Use Bonferroni or Tukey for multiple comparisons.

# Example - School days